IRMA Lectures in Mathematics and Theoretical Physics 3

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From Combinatorics to Dynamical Systems

Journées de Calcul Formel, Strasbourg, 2002

Editors: F. Fauvet and C. Mitschi

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Edited by Vladimir G. Turaev

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From Combinatorics to Dynamical Systems

Journées de Calcul Formel

Strasbourg, March 22–23, 2002

Editors

F. Fauvet
C. Mitschi

Walter de Gruyter · Berlin · New York

Editors

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Preface

This volume presents the proceedings of the Conference "Journées de calcul formel en l'honneur de Jean Thomann", which took place at IRMA in March 2002. This meeting was aimed at bringing together mathematicians working in various areas of Thomann's interests, specifically in dynamical systems, computer algebra, and theoretical physics.

Jean Thomann was born on June 30, 1936, in Strasbourg. He received his mathematical education in Strasbourg where among his professors were Marcel Berger, Jean-Louis Koszul and René Thom. While a research engineer with the IN2P3, Institut National de Physique des Particules, he worked at the Institut de Recherche Nucléaire, at the Centre de Calcul du CNRS in Strasbourg, and at IRMA.

It is important to mention that the present volume has historical ties to the proceedings([1]), ([2]), ([3]) of three pioneering workshops "Computer Algebra and Differential Equations" (*CADE*) held in Grenoble 1988, Cornell 1990, Luminy 1992. These proceedings have become standard references in the area, and as with these classical texts, this volume includes papers of both theoretical and applied nature.

The unifying theme of these proceedings is computer algebra. As for the organization of the material, the book begins with articles of combinatorial or algebraic nature, then proceeds to articles involving more analytical tools.

The book includes a foreword by Jean-Pierre Ramis (in French) followed by nine articles.

The first paper by M. Espie, J.-C. Novelli and G. Racinet is devoted to computations in certain graded Lie algebras related to multiple zeta values. Some of these computations have been implemented in computer algebra systems with an unexpected experimental result leading to a new conjecture. The next four articles study differential systems with irregular singularities from algebraic as well as analytic points of view. The article of E. Corel gives an algebraic and algorithmic treatment of formal exponents at an irregular singularity and provides new and effective methods for computing such exponents and exponential parts for solutions of linear differential systems. The article of J. L. Martins studies irregular meromorphic connections in higher dimensions based on the seminal work of C. Sabbah. Using essentially algebraic techniques, the author obtains results concerning the summability of solutions. The paper of M. A. Barkatou, F. Chyzak and M. Loday-Richaud describes and compares various algorithms concerning the rank reduction of differential systems. It is

[1] *Computer Algebra and Differential Equations*, edited by E. Tournier, Academic Press, 1989

[2] *Differential Equations and Computer Algebra,* edited by M. F. Singer, Academic Press, 1991

[3] *Computer Algebra and Differential Equations*, London Math. Soc. Lecture Note Ser. 193, edited by E. Tournier, Cambridge University Press, 1994

intimately related to questions of classification of irregular connections and computations of differential Galois groups. The article of M. Canalis-Durand presents new formal and numerical algorithms providing estimates of the growth of coefficients for a certain type of divergent series which are ubiquitous in dynamical systems with irregular singularities, such as in Quantum Field Theory.

The next three papers contain effective calculations, formal and numerical, for dynamical systems arising from certain questions in physics. The article of D. Boucher and J.-A. Weil presents computations of non-integrability of Hamiltonian systems, based on the Morales-Ramis theorem in differential Galois theory. The paper of L. Brenig uses the Quasi Polynomial formalism and generalized Lotka–Volterra normal forms to obtain effective integrability for some dynamical systems, with interesting combinatorial consequences. The article of R. Conte, M. Musette and T.-L. Yee involves computations of Padé approximants related to concepts of analytic integrability of equations of physical significance with chaotic behaviour.

The article of J. Della Dora and M. Mirica-Ruse explains a new mathematical formalization of the recent concept of hybrid systems, with ideas and techniques from dynamical systems and automata theory.

This conference was supported by IRMA (Strasbourg), with additional financing by Institut IMAG (Grenoble), Institut Universitaire de France and Université d'Angers, which we thankfully acknowledge. We express our gratitude to Jean-Pierre Ramis for his constant support, to the director and the staff of IRMA for their help during the Conference, to Vladimir Turaev, editor of the IRMA Lectures in Mathematics and Theoretical Physics for kindly suggesting to publish these proceedings, and to the mathematics editor of de Gruyter.

Strasbourg, September 2003 *Frédéric Fauvet and Claude Mitschi*

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Jean Thomann

Hommage à Jean Thomann

En préambule à ce volume d'hommage à Jean Thomann, je voudrais évoquer quelques souvenirs de notre aventure commune en équations différentielles et en calcul formel.

A la fin des années 1975, j'avais l'idée vague et optimiste de faire des "calculs numériques explicites" sur les équations différentielles ordinaires algébriques, dans le cas singulier irrégulier. Une première tentative dans ce sens, par Jean Martinet et moi-même (avec une totale inexpérience…) dans le cas des noeuds-cols de Ricatti, s'était soldée par un échec, le birapport cherché passant d'un ordre 1 à l'ordre 100 selon le moment où nous lancions le calcul… J'avais au moins découvert l'extrême instabilité numérique du problème (ce qui avait motivé notre recherche d'une solution théorique). J'ai parlé de mes problèmes à Jean Françon, qui m'a conseillé de prendre contact avec Jean Thomann. C'est ainsi que Jean a commencé à fréquenter nos séminaires, à poser des questions amorçant des discussions, qui au début surprenaient un peu, je crois, le public habituel. Le premier problème sur lequel a débuté notre collaboration fut celui de la qualité numérique de la sommation des séries de facultés (factorial series). Tout le monde pense qu'elle est mauvaise, mais une petite remarque de Nörlund disant le contraire m'avait intrigué. Nous avons montré dans un article commun que Nörlund avait raison, mais l'explication théorique manque toujours à ce jour… Dans la foulée (et dans la ligne de Laguerre et Borel) nous avons eu l'idée de mettre en oeuvre, dans le cadre différentiel, des techniques d'approximation rationnelle du style Padé, que Jean connaissait bien et que j'ignorais presque complètement. Comme Jean était proche du groupe de l'IMAG[1] de Grenoble (autour de N. Gastinel) il m'a proposé de discuter de tout cela avec Jean Della Dora. Jean Della Dora avait fait sa thèse sur les approximants de Padé et Bernard Malgrange lui avait proposé comme sujet de deuxième thèse la décomposition formelle des opérateurs d'ordre un. (La deuxième thèse était souvent bien utile…) Nous avons ainsi amorcé des discussions à trois, mais il manquait un acteur, ou plutôt une actrice, pour faire démarrer vraiment notre projet de "solver" d'EDO (équations différentielles ordinaires). Tout s'est concrétisé en quelques heures à Grenoble, dans une discussion à laquelle participait Evelyne Tournier qui rentrait juste de Salt Lake City avec "dans ses bagages" le processus formel *Reduce*. Notre projet commun Grenoble-Strasbourg pouvait ainsi démarrer, centré sur le calcul formel. Toute une activité de calcul formel s'est alors développée à Grenoble autour de ce projet (thèse d'Evelyne, travaux de Claire Di Crescenzo) et des difficultés qu'il rencontrait (l'une des motivations de la thèse de Dominique Duval fut le problème de l'exécution en calcul formel des algorithmes de type polygone

[1] Institut d'Informatique et Mathématiques Appliquées de Grenoble

de Newton itéré, nécessitant l'empilement d'équations algébriques). Du côté strasbourgeois nous étions toujours intéressés par les aspects numériques et nous avons eu l'idée de la nécessité de développer une interface formel-numérique. Le premier prototype (Reduce-Fortran) a été bricolé par Jean Thomann et James Davenport. Il est vraisemblable que ce fut le premier (ou l'un des tout premiers) dans l'histoire. Les choses ont bien changé... Dans les premières applications, l'interface tournait avec un transport matériel de bandes magnétiques entre Grenoble et Strasbourg... A cette époque Jean Thomann s'est lancé avec son énergie et son enthousiasme habituels dans la diffusion du calcul formel auprès des collègues de diverses disciplines scientifiques et des ingénieurs. Il a ainsi organisé des cours d'initiation qui étaient très convaincants et qui nous ont permis de nouer divers contacts scientifiques. Une retombée de cette action, par exemple, est la solution par A. Gorius (Rhône-Poulenc) d'un problème de chromatographie industrielle (l'accélération de "convergence" de séries – dont on ne savait pas au demeurant si elles convergeaient ou non – permettant un traitement en temps réel sur une petite machine).

Avec sa générosité habituelle Jean Thomann encadrait de nombreux projets d'étudiants et aidait beaucoup les jeunes chercheurs de Strasbourg et de Grenoble. Il a joué un rôle central dans la thèse de Françoise Richard-Jung. Cette thèse développait les interfaces formel-graphique(couleur)-numérique. Matériellement, elle a bénéficié des nombreuses heures de calcul gratuites et du matériel graphique performant du Centre de calcul de Cronenbourg. Il est inutile de dire que la gentillesse de Jean Thomann et sa force de persuasion ont contribué à obtenir ces moyens... Plus fondamentalement, Jean a joué un rôle important dans la structuration et l'élaboration de l'interface. Tout cela a abouti à la réalisation du "solver" prototype d'EDO nommé DESIR, formel-numérique-graphique (Grenoble-Strasbourg), dont nous étions assez fiers... J'ai une certaine nostalgie de cette époque. Nous formions une petite équipe unie par beaucoup d'enthousiasme et d'optimisme, avec pas mal de naïveté parfois, et beaucoup d'amitié. L'un des moments les plus marquants fut une semaine de travail intensif organisée par Jean Thomann dans un VVF[2] des Vosges. C'était un peu spartiate et l'outil de base était un tableau noir d'enfant. Sur ce tableau ont été esquissés beaucoup de projets qui ont pris corps durant les années qui ont suivi.

Plusieurs raisons poussaient, à partir de là, à passer à la vitesse supérieure. Entre autres le souhait de sortir de la phase expérimentale pour le logiciel, mais aussi les progrès théoriques importants qui venaient d'être réalisés, en particulier du côté de la multisommabilité et de la théorie de Galois différentielle.

Le point de départ de la nouvelle période est sans doute la première conférence internationale CADE (Computer Algebra and Differential Equations) organisée par Evelyne Tournier près de la Grande Chartreuse. Elle a permis de nouer de nouveaux liens au niveau international (Allemagne, Belgique, U.S.A.). Ensuite ont émergé les deux *Working groups* CATHODE du projet européen ESPRIT (de 1992 à 1995 puis de 1997 à 2000); Jean Thomann était membre de leur conseil scientifique. Sa

[2]Villages de Vacances en France

participation a été un élément central, sur le plan scientifique et celui de l'animation bien sûr, mais aussi pour l'ambiance de travail : sa gentillesse et son sens de l'amitié ont été souvent précieux pour désamorcer conflits ou tensions inévitables dans un groupe de taille importante, international et formé de composantes issues de cultures scientifiques assez différentes. (Nous étions loin de la structure plutôt familiale des débuts…)

Il me faudrait encore parler du rôle de Jean dans DESIR2, de son investissement dans les travaux en logiciel SCRATCHPAD développés à Strasbourg et Grenoble (par C. Arnold, C. Di Crescenzo, F. Naegelé-Marotte, F. Richard-Jung), de sa récente collaboration avec F. Fauvet autour de la résurgence…

Merci Jean, pour toutes ces années passionnantes aux confins des mathématiques et de l'informatique, et surtout pour ta chaleur humaine, ton ouverture et ton optimisme constants.

Jean-Pierre Ramis

Formal computations about multiple zeta values

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Abstract. In this paper, we compute the dimensions of the graded Lie algebra $\mathfrak{dmr}_0$ (introduced by Racinet in [Rac00]) up to a given degree. These results prove that, up to this degree, the Lie algebra $\mathfrak{dmr}_0$ is free over one element in each odd degree greater than or equal to 3, this element being the one introduced by Drinfeld in [Dri90] and belonging to his Lie algebra $\mathfrak{grt}_1$. In particular, this proves, up to the same degree, that the double shuffle and regularization relations imply the associators relations, and that the formal dimension conjecture of Zagier is true.

Keywords. Shuffle products, multiple zeta values, zeta function, associators, free Lie algebras, quasi-symmetric functions, conjecture of Deligne–Drinfeld, Grothendieck–Teichmüller group.

1 Introduction

The numbers we call multiple zeta values (MZVs, for short) are values at positive integers of the following generalization of Riemann's zeta function:

$$\zeta(s_1, \ldots, s_r) := \sum_{n_1 > n_2 > \cdots > n_r > 0} 1/n_1^{s_1} n_2^{s_2} \ldots n_r^{s_r} \tag{1.1}$$

For such sequences of positive integers, this series will be finite if and only if $s_1 \neq 1$. There has been a big amount of research around these numbers during the past decade, either theoretical or computational, since Zagier's talk at the first ECM [Zag94].

Many combinatorial problems related to quantum algebra become simpler if the base field contains the MZVs. This happens because their full collection is the only explicitly defined solution of some relevant infinite system of equations. This is an obstruction to let the computers work on these problems, since the transcendance properties of the MZVs seem to be out of reach. There are however precise transcendance conjectures about the MZVs. For instance, Zagier conjectured that the dimensions d_n of the $\mathbb{Q}$-vector subspace of $\mathbb{R}$ spanned by the MZVs of weight $n = s_1 + \cdots + s_r$ should satisfy the Fibonacci-like $d_n = d_{n-2} + d_{n-3}$.

In this paper, we will focus on two infinite systems of algebraic relations between the MZVs: the "regularized double shuffle" relations (DMR)[1] and the associators relations. The latter already led to an impressive amount of applications [EK96], [BN97], [Piu95], [Car93], [Kon99], [Tam], and the former are somewhat simpler. Both are conjecturally equivalent. One of the goals of this paper is to check that the DMR relations imply the associators', up to a given weight. Our method relies on results of the third author [Rac01], [Rac02], that allow to perform this task without writing down the relations of the associators. Anyway, let us sketch for completeness some parts of this theory.

Let a and b be two formal non-commuting symbols. In the seminal paper [Dri91], Drinfeld introduced the associators, which he defined as the Lie exponentials Φ built on a, b which satisfy:

$$\Phi(b,a)\Phi(a,b) = 1 \quad \text{(duality, or 2-cycle)} \tag{1.2}$$

$$e^{\lambda a}\Phi(c,a)e^{\lambda c}\Phi(b,c)e^{\lambda b}\Phi(a,b) = 1 \quad \text{(hexagonal, or 3-cycle)}, \tag{1.3}$$

where c stands for $-a-b$ and λ is a scalar, together with a third equation, called *pentagonal*, or 5-*cycle*, which is more complicated. These equations are equivalent to an infinite system of constraints between the coefficients of Φ. Using the monodromy properties of the formal Knizhnik–Zamolodchikov differential equations, Drinfeld built an associator Φ_{KZ} with complex coefficients. Although he proved the existence of rational associators, none has so far been *constructed*. It was later understood that Φ_{KZ} is some kind of non-commutative generating series encoding all the MZVs (see, e.g., [LM96], [HPV99]). We can therefore understand Drinfeld's equations as a system of relations between them.

Drinfeld also introduced two infinite matrix groups acting on associators, the Grothendieck–Teichmüller group GT_1 and its "graded" version GRT_1. The Lie algebra $\mathfrak{grt}_1$ of GRT_1, also known as Ihara's "stable derivation algebra" [Iha92] has also very interesting relationships with the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ (see [Iha91] for an excellent review of these topics).

The third author introduced a graded Lie algebra $\mathfrak{dmr}_0$ playing the role of $\mathfrak{grt}_1$ in the DMR setting. In this paper, we prove by computer computation the inclusion of $\mathfrak{dmr}_0$ in $\mathfrak{grt}_1$.[2] This is easily seen to mean that the DMR relations imply the associators

[1] This stands for the french "Double mélange et régularisation".

[2] Here and below, all results are valid up to weight 19.

relations. For fastening up the computation, we use insights from Drinfeld's conjectural description of $\mathfrak{grt}_1$, which we call Deligne–Drinfeld conjecture for convenience. Indeed, we prove it, with $\mathfrak{grt}_1$ being replaced by $\mathfrak{dmr}_0$. This gives for $\mathfrak{grt}_1$ still one half of the conjecture – the freeness part. The full result about $\mathfrak{dmr}_0$ also proves the counterpart of Zagier's conjecture for formal consequences of the DMR system.

Some related computations were done by the Lille team up to weight 10 [HP00], then 12 [BJOP98]. They computed a Gröbner basis of the DMR system[3] and checked that Zagier's conjecture is true for DMR, up to this weight. The price to pay for the method we use here is that it provides all relations between MZV's deduced from DMR only up to some rational multiple of $\zeta(2)^n$ in weight $2n$.

1.1 Structure of the paper

We first quickly review the first properties of the MZVs and the double shuffle and regularization relations (see Section 2). We then introduce the algebra of formal MZVs and its dual variety DMR (see Section 3). Next, we give the definition of the Lie algebra $\mathfrak{dmr}_0$ and state the main theorem of [Rac02]. We then discuss the conjectural description of Drinfeld's $\mathfrak{grt}_1$ and explain some reductions (see Section 4). Finally, we describe the implementation of the computation in Section 6.

1.2 Acknowledgments

The authors would like to thank Tobias Weingartner and the Department of Computer Science of University of Alberta, Canada, for gracious use of their computing resources.

2 Multiple zeta values and double shuffle relations

Throughout this paper, $\mathbb{K}$ denotes a commutative ring with unit which contains $\mathbb{Q}$, the field of rationals, $\mathbb{K}\langle A\rangle$ denotes the algebra of non commutative polynomials over the set A, and $\mathbb{K}\langle\langle A\rangle\rangle$ denotes the algebra of formal series over A. We will denote by A^* the set of words on A; the empty word will be denoted ϵ.

Finally, the letter S will represent the set of ordered sequences of strictly positive integers (also known as compositions) and S^{cv} the subset of sequences that do not begin with 1.

[3] Seemingly, they use only a subset of DMR. The same argument as in the present article fills the gap.

2.1 First shuffle relations

The first shuffle relations come from some well-known iterated integral representation of MZVs (multiple zeta values) that will not be discussed in this paper (see, *e.g.*, [Gon00]). We will only present how these relations translate in the multiple zeta values context.

2.1.1 The usual shuffle product. All the material presented in this paragraph can be found with much more details in [Reu93].

Let X be the alphabet $X = \{x_0, x_1\}$. Let a and b be two elements of X^* written as $a = a_1 a'$ and $b = b_1 b'$, where a_1 and b_1 belong to X. Then one recursively defines the *shuffle product* of a and b as

$$(a_1 a') \sqcup\!\sqcup (b_1 b') = a_1 (a' \sqcup\!\sqcup b_1 b') + b_1 (a_1 a' \sqcup\!\sqcup b'), \tag{2.1}$$

with $\epsilon \sqcup\!\sqcup x = x \sqcup\!\sqcup \epsilon = x$, for every x in X^*.

For example, one can check that

$$x_0 x_1 \sqcup\!\sqcup x_0 x_0 x_1 = x_0 x_1 x_0 x_0 x_1 + 3 x_0 x_0 x_1 x_0 x_1 + 6 x_0 x_0 x_0 x_1 x_1. \tag{2.2}$$

It is known that the shuffle product endows $\mathbb{K}\langle X\rangle$ with a free commutative algebra structure.

We will have later to consider two subalgebras of $\mathbb{K}\langle X\rangle$. The first one is the algebra $\mathbb{K}\langle X^c\rangle$ spanned by X^c, the set of words in X that do not end with an x_0 (it is closed since the shuffle of two words that do not end with an x_0 is a linear combination of such words). The second one is the algebra $\mathbb{K}\langle X^{cv}\rangle$, where X^{cv} is the set of words in X that do not end with an x_0 and do not begin with an x_1.

One can endow the usual concatenation algebra $(\mathbb{K}\langle X\rangle, \cdot)$ with a Hopf algebra structure, the coproduct $\Delta \colon \mathbb{K}\langle X\rangle \to \mathbb{K}\langle X\rangle \otimes \mathbb{K}\langle X\rangle$ being given as an algebra morphism by

$$\Delta x_i = 1 \otimes x_i + x_i \otimes 1, \quad i \in \{0, 1\}.$$

There is a well-known duality between Δ and $\sqcup\!\sqcup$ that will be used implicitly in Section 3.

2.1.2 Connection with the multiple zeta values. Let us begin with some notation. If x belongs to X^c, one can write x as

$$x = x_0^{s_1-1} x_1 x_0^{s_2-1} x_1 \ldots x_0^{s_r-1} x_1, \tag{2.3}$$

where $\boldsymbol{s} = (s_1, \ldots, s_r)$ is an element of S. So there exists a natural bijection between X^c and S: the word x written in Equation (2.3) is associated with the sequence $\boldsymbol{s}$. Notice that this correspondence provides a bijection between X^{cv} and S^{cv}, as well.

Thanks to this bijection, one can define $\boldsymbol{s} \sqcup\!\sqcup \boldsymbol{s}'$ with $\boldsymbol{s}$ and $\boldsymbol{s}'$ in S, as the shuffle of the two corresponding elements of X^c, expressed as a formal sum of elements of S.

With this notation, one can now state the *shuffle relations of the first kind* as

$$\forall \boldsymbol{s}, \boldsymbol{s}' \in S^{cv}, \quad \zeta(\boldsymbol{s})\zeta(\boldsymbol{s}') = \zeta(\boldsymbol{s} \;\sqcup\!\sqcup\; \boldsymbol{s}'), \tag{2.4}$$

where ζ is extented by linearity to formal sums.

For example, if one considers $\boldsymbol{s} = (2)$ and $\boldsymbol{s}' = (3)$, the two equations (2.2) and (2.4) lead to

$$\zeta(2)\zeta(3) = \zeta(2,3) + 3\zeta(3,2) + 6\zeta(4,1). \tag{2.5}$$

Remark 2.1. For $\boldsymbol{s} = (s_1, \ldots, s_r)$, one defines the *weight* $w(\boldsymbol{s})$ of $\boldsymbol{s}$ as $\sum s_i$. One can then easily see that the usual shuffle product is a homogeneous operation with respect to the weight on sequences. Notice that this weight is exactly the total degree of the corresponding word in X^*, whereas the length r is the partial degree in x_1.

2.2 The second shuffle relations

The second shuffle relations can be understood as the product of monomial quasi-symmetric functions, as defined in [Ges84] and extensively studied in [GKL+95], [MR95]. Indeed, the MZVs appear as special values of these functions.

2.2.1 The second shuffle product. For notation safety, we shall (tautologically) replace finite sequences by words built on an infinite alphabet of non-commuting letters $Y = \{y_1, y_2, \ldots\}$.

For any words $u, v \in Y^*$ and positive integers p, q, the product $*$ is given by the following recursive formula:

$$\begin{aligned} (y_p u) * (y_q v) &= (u * y_q v) + y_q(y_p u * v) + (y_{p+q})(u * v), \\ \epsilon * u &= u * \epsilon = u, \end{aligned} \tag{2.6}$$

where u and v are any words in Y^* and p, q are positive integers. For example, we have

$$y_2 * y_3 = y_2 y_3 + y_3 y_2 + y_5. \tag{2.7}$$

Remark 2.2. Moreover, one can endow the *concatenation* algebra $(\mathbb{K}\langle Y\rangle, \cdot)$ with a bialgebra structure with coproduct Δ_* defined as

$$\forall n \in \mathbb{N}, \quad \Delta_*(y_n) = \sum_{k+l=n} y_k \otimes y_l, \quad \text{with } y_0 := 1. \tag{2.8}$$

It is then known that the coproduct is dual to the product $*$ (see [MR95]).

2.2.2 Connection with the multiple zeta values. As we mentioned before, the MZVs are specializations of quasi-symmetric functions and thus inherit the following property, as was extensively studied in [Hof95], [Hof97]:

$$\forall \boldsymbol{s}, \boldsymbol{s}' \in S^{cv}, \; \zeta(\boldsymbol{s})\zeta(\boldsymbol{s}') = \zeta(\boldsymbol{s} * \boldsymbol{s}'), \tag{2.9}$$

where the product $*$ is interpreted on sequences.

This can also be proved by direct induction on the definition in Formula (1.1).

In the situation of the example (2.5), we now have, by Equations (2.7) and (2.9):

$$\zeta(2)\zeta(3) = \zeta(2,3) + \zeta(3,2) + \zeta(5). \tag{2.10}$$

Remark 2.3. As in the case of the first shuffle product, one can easily see that the second shuffle product is a homogeneous operation for the weight on sequences.

These examples also illustrate that the similitudes in the combinatorial descriptions of the two kinds of shuffle relations do not prevent them from being quite incompatible. Actually, the evidence of existence of the MZVs is the only known proof of the fact that the two shuffle relations do not contradict each other. The same remark holds for associators.[4]

2.3 Regularization process

It is convenient to give sense to divergent MZVs (*e.g.* $\zeta(1)$) with respect to each shuffle product, so that either Equation (2.4) or Equation (2.9) still holds. The main reason is that it would allow us to define generating series of *all* MZVs that would be by construction group-like elements in the Hopf algebras previously defined and thus would allow us to use powerful algebraic tools. For complete explanations about this part, see, *e.g.*, [Rac00], [Hof97], [HP00]. The process that associates a finite value with each divergent MZV is called the *regularization process*.

In the sequel, we denote by $(\Phi|w)$, where w belongs to A^* and $\Phi \in \mathbb{K}\langle\langle A\rangle\rangle$, the coefficient of w in Φ. We extend this definition by linearity on the right to any element of $\mathbb{K}\langle A\rangle$.

2.3.1 Regularization of the usual shuffle product.

Proposition 2.4 ([HPV98]). *There exists a unique series $\Phi_{\sqcup\!\sqcup}$ in $\mathbb{R}\langle\langle X\rangle\rangle$ satisfying*

$$\Delta(\Phi_{\sqcup\!\sqcup}) = \Phi_{\sqcup\!\sqcup} \otimes \Phi_{\sqcup\!\sqcup} \tag{2.11}$$

such that the coefficient of x_0 and x_1 are zero and the coefficient of every element x of X^{cv} is equal to $\zeta(\mathbf{s})$ where $\mathbf{s}$ is the element of S associated with x.

This proposition allows us to compute $(\Phi_{\sqcup\!\sqcup}|w)$ for a given word w, even if it begins with x_1 and/or ends with x_0 for $\Phi_{\sqcup\!\sqcup}$ being a Lie exponential is equivalent to the relation

$$\forall u, v \in X^*, \quad (\Phi_{\sqcup\!\sqcup}|(u \sqcup\!\sqcup v)) = (\Phi_{\sqcup\!\sqcup}|u)(\Phi_{\sqcup\!\sqcup}|v), \tag{2.12}$$

and is also equivalent to $\Phi_{\sqcup\!\sqcup}$ being a group-like element, that is

$$\Delta\Phi_{\sqcup\!\sqcup} = \Phi_{\sqcup\!\sqcup} \otimes \Phi_{\sqcup\!\sqcup}. \tag{2.13}$$

[4]For associators, there is another such evidence, coming from their ℓ-adic counterparts.

For example, one computes that

$$(\Phi_{ш}|x_1x_1) = 0, \tag{2.14}$$

since $x_1 \sqcup\!\sqcup\, x_1 = 2x_1x_1$. Let us mention that there exists a simple formula given by Cartier (see [Car00]), to directly compute each coefficient of $\Phi_{ш}$ using the Taylor expansion.

It is now well known that $\Phi_{ш}$ and Drinfeld's Φ_{KZ} series are equal, if one identifies a with x_0 and b with $-x_1$.

2.3.2 Regularization of the second shuffle product.

Proposition 2.5. *There exists a unique series Φ_* in $\mathbb{R}\langle\langle Y\rangle\rangle$ satisfying*

$$\Delta_*(\Phi_*) = \Phi_* \otimes \Phi_* \tag{2.15}$$

such that the coefficient of y_1 is zero and the coefficient of an element y of Y^{cv} is equal to $\zeta(\boldsymbol{s})$ where $\boldsymbol{s}$ is the element of S associated with y.

This last proposition allows us to compute $(\Phi_*|w)$ for a given word w, even if it begins with y_1 for Φ_* being a group-like element is equivalent to the relation

$$\forall u, v \in Y^*, \quad (\Phi_*|(u * v)) = (\Phi_*|u)(\Phi_*|v). \tag{2.16}$$

For example, one computes that

$$(\Phi_*|y_1y_1) = -\frac{1}{2}(\Phi_*|y_2) = -\frac{\pi^2}{12}, \tag{2.17}$$

since $y_1 * y_1 = 2y_1y_1 + y_2$. Again, let us mention that there exists a simple formula to directly compute each coefficient of Φ_* using the Taylor expansion that was given by Cartier (see [Car00]).

Notice that both regularizations do not give the same result for (1, 1) (see Equations 2.14 and 2.17). Still, some comparison is possible. This is precisely what the last relation (see Proposition 2.6) provides.

2.3.3 Comparison between both regularizations. The third system of relations between MZVs can be expressed as a comparison formula between the series $\Phi_{ш}$ and Φ_* (see, *e.g.*, [Rac02] for more details, including historical remarks):

Proposition 2.6. *The series $\Phi_{ш}$ and Φ_* are deduced one from the other by the equation*

$$\Phi_* = \exp\left(\sum_{n\geqslant 2} \frac{(-1)^{n-1}}{n}\zeta(n)y_1^n\right)\Pi_Y(\Phi_{ш}), \tag{2.18}$$

where Π_Y stands for the linear projection from $\mathbb{R}\langle\langle X\rangle\rangle$ to $\mathbb{R}\langle\langle Y\rangle\rangle$ that sends each word ending with x_0 to 0 and each $x = x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1$ to $y_{s_1} \ldots y_{s_r}$.

The first new relation arising from this proposition is

$$\zeta(3) = \zeta(2, 1), \tag{2.19}$$

which dates back to Euler.

Indeed, one can easily check that the coefficient of $y_2 y_1$ on the left-hand side of Equation (2.18) is $2\zeta(2, 1)$, whereas its coefficient on its right-hand side is $\zeta(3) + \zeta(2, 1)$.

Let us notice that this third system of relations implies the Hoffman system of relations (see [Hof97], [Rac00]).

Remark 2.7. As in the case of the first and the second shuffle product, one can see that the relations summarized by Equation (2.18) are homogeneous for the weight on sequences. This compatibility of both shuffle products and regularization relations with the weight enables us to study the relations between MZVs weight by weight.

3 Formal MZVs

It is a conjecture that the double shuffle and regularization relations generate all possible algebraic relations with rational coefficients between the MZVs. This conjecture is far out of reach at the present time: it would imply[5] for example the algebraic independence of the $\zeta(2n+1)_{n\geqslant 0}$. Nevertheless, this conjecture suggests to focus on the *formal* consequences of the three above relations.

3.1 The dual variety

One natural way to address this problem would be to consider the algebra PZF of "formal MZVs", *i.e.*, algebra of unknowns $\overline{\zeta}(s)_{s\in S}$ satisfying relations (2.4), (2.9), and (2.18). For many reasons, the most obvious being the compact form that our three relations take on generating series, we shall rather work with the following dual object.

Definition 3.1. For any commutative ring $\mathbb{K}$, such that $\mathbb{Q} \subset \mathbb{K}$, let $\mathsf{DMR}_\lambda(\mathbb{K})$ be the set of elements Φ of $\mathbb{K}\langle\langle X\rangle\rangle$ such that

$$(\Phi|1) = 1 \tag{3.1}$$

$$(\Phi|x_0) = (\Phi|x_1) = 0 \tag{3.2}$$

$$(\Phi|x_0x_1) = \lambda \tag{3.3}$$

[5] This is actually an easy consequence of Theorem 3.4 below.

$$\Delta\Phi = \Phi \mathbin{\widehat{\underset{\mathbb{K}}{\otimes}}} \Phi \tag{3.4}$$

$$\Delta_*\Phi_* = \Phi_* \mathbin{\widehat{\underset{\mathbb{K}}{\otimes}}} \Phi_*, \tag{3.5}$$

where

$$\Phi_* := \exp\left(\sum_{n\geqslant 2} \frac{(-1)^{n-1}}{n}(\Pi_Y(\Phi)|y_n)y_1^n\right)\Pi_Y(\Phi) \tag{3.6}$$

The results of the previous sections now read: $\Phi_ш \in \mathsf{DMR}_{\zeta(2)}(\mathbb{R})$. The relevance of this definition to our problem is due to the following easy fact:

Proposition 3.1. *There is a natural bijection between* $\mathsf{DMR}_\lambda(\mathbb{K})$ *and the set of algebra homomorphisms from* PZF *to* $\mathbb{K}$ *sending* $\overline{\zeta}(2)$ *to* λ.

Here "natural" means that it is compatible with maps induced on both sides by algebra morphisms $\mathbb{K} \to \mathbb{K}'$. In other words, $\mathsf{DMR}_\lambda(\mathbb{K})$ is the set of all generating series encoding collections of elements of $\mathbb{K}$ satisfying our three relations, λ playing the role of $\zeta(2)$. By general abstract considerations (Yoneda's Lemma), it is equivalent to study the functor $\mathbb{K} \mapsto \mathsf{DMR}(\mathbb{K})$ in place of PZF.

This change of perspective also makes the comparison with Drinfeld associators easier to perform.

3.2 The Lie algebra

Many properties of DMR can be reduced to the tangent space $\mathfrak{dmr}_0$ at 1 of DMR_0, as it follows from Theorem 3.4 below. Informally speaking, to get the defining equations of this tangent space, one only needs to linearize equations (3.1)–(3.6). Here is a direct definition:

Definition 3.2. Let $\mathfrak{dmr}(\mathbb{K})$ be the set of elements ψ of $\mathbb{K}\langle\langle X\rangle\rangle$ such that

$$\begin{aligned} (\psi|1) &= 1 && (3.7)\\ (\psi|x_0) &= (\psi|x_1) = 0 && (3.8)\\ \Delta\psi &= \psi\otimes 1 + 1\otimes\psi && (3.9)\\ \Delta_*\psi_* &= \psi_*\otimes 1 + 1\otimes\psi_*, && (3.10)\end{aligned}$$

where

$$\psi_* := \Pi_Y(\psi) + \sum_{n\geqslant 2}\frac{(-1)^{n-1}}{n}(\Pi_Y(\psi)|y_n)y_1^n \tag{3.11}$$

We denote by $\mathfrak{dmr}_0(\mathbb{K})$ the subset of $\mathfrak{dmr}(\mathbb{K})$ of elements ψ such that $(\psi|x_0x_1) = 0$.

Remark 3.3. Notice that all spaces $\mathfrak{dmr}_0(\mathbb{K})$ can be reduced to $\mathfrak{dmr}_0(\mathbb{Q})$ (or $\mathfrak{dmr}_0$, for short) since $\mathfrak{dmr}_0(\mathbb{K}) = \mathfrak{dmr}_0 \otimes \mathbb{K}$, by linearity and homogeneity of the above equations.

The definition of the tangent spaces find its use in the next theorem.

Theorem 3.4 ([Rac01], [Rac02]). *Let G and H be elements of $\mathsf{DMR}_\lambda(\mathbb{K})$. There exists a unique ψ in $\mathfrak{dmr}_0(\mathbb{K})$ such that $\exp(s_\psi)G = H$, where s_ψ is the sum of the right multiplication by ψ with the derivation D_ψ sending x_0 to $[\psi, x_0]$ and x_1 to 0.*

For any $\lambda \in \mathbb{Q}$, the set $\mathsf{DMR}_\lambda(\mathbb{Q})$ is not empty.

Ihara's bracket, $\langle \psi_1, \psi_2 \rangle := s_{\psi_1}(\psi_2) - s_{\psi_2}(\psi_1)$ endows $\mathfrak{dmr}_0$ with a Lie algebra structure.

We will explain in detail in Section 4 how to use this theorem to prove that the regularized double shuffle relations imply a given system of relations. Our main example being the system related to the associators of Drinfeld, we shall first recall some of their properties.

4 Some reductions

4.1 Associators and double shuffles

As we mentioned in the introduction, one aim of this paper is to compare the systems of relations coming from the double shuffle and from the associators. We are going to prove that, up to a certain weight, the DMR relations imply the associators relations.

Let $\mathsf{Ass}_\lambda(\mathbb{K})$ denote the set of associators with parameter λ in Equation (1.3) and with coefficients in $\mathbb{K}$. Let $\mathfrak{grt}_1$ be the tangent space of Ass_0 at 1. Forgetting the weight restriction, our aim restates as proving the inclusion of $\mathsf{DMR}_\lambda(\mathbb{K})$ in $\mathsf{Ass}_\lambda(\mathbb{K})$, for all $\mathbb{K}$ and $\lambda \in \mathbb{K}$.

Actually, Theorem 3.4 was inspired by Drinfeld's results.

Theorem 4.1 (Drinfeld [Dri91], Propositions 5.5 and 5.9). *Fix $\lambda \in \mathbb{K}$ and let G and H be elements of $\mathsf{Ass}_\lambda(\mathbb{K})$. There is a unique $\psi \in \mathfrak{grt}_1(\mathbb{K})$ such that $\exp(s_\psi)G = H$. Rational associators do exist,* i.e., $\mathsf{Ass}_1(\mathbb{Q}) \neq \varnothing$.

Another common feature of Ass and DMR is the homogeneity of their defining equations. This can be stated as follows: let $[\mu]\Phi$ denote the series deduced from Φ by multiplication of the homogeneous component of weight n by μ^n. Then $\Phi \in \mathsf{Ass}_\lambda(\mathbb{K})$ implies $[\mu]\Phi \in \mathsf{Ass}_{\lambda\mu}(\mathbb{K})$ while $\Phi \in \mathsf{DMR}_\lambda(\mathbb{K})$ implies $[\mu]\Phi \in \mathsf{DMR}_{\lambda\mu^2}(\mathbb{K})$, because the weight of x_0x_1 is 2.

Now, thanks to Theorems 3.4 and 4.1, the comparison of DMR and Ass boils down to the Lie algebras $\mathfrak{dmr}_0$ and $\mathfrak{grt}_1$: let us assume for instance we have the inclusion $\mathfrak{dmr}_0 \subset \mathfrak{grt}_1$. Recall that $\Phi_{\sqcup\!\sqcup}$ belongs to *both* $\mathsf{DMR}_{\zeta(2)}(\mathbb{C})$ and $\mathsf{Ass}_{i\pi}(\mathbb{C})$.

Applying $[1/i\pi]$, we thus get a common element of $\mathsf{DMR}_{-1/6}(\mathbb{C})$ and $\mathsf{Ass}_1(\mathbb{C})$. Using both theorems and the Lie algebra inclusion, we get the inclusion of $\mathsf{DMR}_{-1/6}(\mathbb{C})$ in $\mathsf{Ass}_1(\mathbb{C})$, whence in particular of $\mathsf{DMR}_{-1/6}(\mathbb{Q})$ in $\mathsf{Ass}_1(\mathbb{Q})$. Now let Ψ belong to $\mathsf{DMR}_{-1/6}(\mathbb{Q})$. For any $\lambda \in \mathbb{K}$, we thus have a common element $[\lambda]\Psi$ in $\mathsf{DMR}_{-\lambda^2/6}(\mathbb{K})$ and $\mathsf{Ass}_\lambda(\mathbb{K})$. Another use of both theorems completes the task:

$$\forall \mathbb{K},\ \forall \lambda \in \mathbb{K},\ \mathsf{DMR}_{\frac{-\lambda^2}{6}}(\mathbb{K}) \text{ is included in } \mathsf{Ass}_\lambda(\mathbb{K}). \tag{4.1}$$

If we assume our inclusion up to some weight, the obvious homogeneity properties of the s_ψ's ensure that this implication still holds up to the given weight.

The same argument will obviously work for any couple of systems of $\mathbb{Q}$-algebraic relations between the MZVs, provided they satisfy adapted versions of Theorems 3.4 and 4.1, if one already has the inclusion at the Lie algebra level. This way, one obtains that the "motivic" relations (see [DG]) imply both DMR and the associators' relations, even though we cannot write them down explicitly. It is enough to know the relevant Lie algebra, which happens to be generated by the special elements we shall now introduce.

4.2 Drinfeld's indecomposables

In the last page of [Dri91], Drinfeld relates the following question to conjectures stated by Deligne:

Question 4.1. *Is $\mathfrak{grt}_1$ free as a Lie algebra, with exactly one homogeneous generator of each odd weight different from* 1?

Moreover, he gave a very simple construction of indecomposable elements of $\mathfrak{grt}_1(\mathbb{C})$ with the expected weights. Their definition reads as follows: it's not hard to show the equality of $\mathsf{Ass}_{-\lambda}(\mathbb{K})$ and $\mathsf{Ass}_\lambda(\mathbb{K})$. There is therefore an element ψ of $\mathfrak{grt}_1(\mathbb{C})$ such that $\exp(s_\psi)$ maps Φ_{KZ} to $[-1]\Phi_{\mathrm{KZ}}$. Define ψ_n to be the weight n part of ψ. It is non zero if n is odd and at least 3. In that case, the coefficient of $y_n = x_0^n x_1$ in ψ_n is $\zeta(n)$, up to a nonzero rational. This, together with the indecomposability of ψ_n's, follows from the obvious homogeneity of the map $(u, v) \mapsto s_u(v)$ with respect to the partial degrees in x_0 and x_1.

As a direct consequence of Theorem 3.4, these elements belong as well to $\mathfrak{dmr}_0(\mathbb{C})$. As the rank of a linear system is invariant under field extensions, we have as well a collection $(\sigma_{2n+1})_{n\geqslant 0}$ of elements of $\mathfrak{dmr}_0(\mathbb{Q}) \cap \mathfrak{grt}_1(\mathbb{Q})$ in which $x_0^{2n}x_1$ appears with coefficient 1. In particular, the freeness part of the question of Drinfeld can be asserted as well in $\mathfrak{dmr}_0$, while it would be sufficient to prove that $\mathfrak{dmr}_0$ is generated by the collection of the σ_{2n+1} to get the inclusion of $\mathfrak{dmr}_0$ in $\mathfrak{grt}_1$.

5 Experimental protocol

Following those ideas, we first check that the dimension of $\mathfrak{dmr}_0$ is smaller than the dimension of a free algebra with one generator in each component of odd weight greater than or equal to 3. This part is done by Gaussian pivot over huge matrices.

We then get a set of elements that span $\mathfrak{dmr}_0$. In each odd weight, we select one element of this set, in which $x_0^{2n}x_1$ appears with non-zero coefficient and call it τ_{2n+1}. It then remains to check that these elements generate a free algebra up to the given weight.

We detail these two parts with some technical improvements in the next two subsections.

5.1 Computation of an upper bound of Dim($\mathfrak{dmr}_0$)

We first present the algorithm used to compute an upper bound of $\mathrm{Dim}(\mathfrak{dmr}_0)$ and next justify it. Since we are only interested in the computation of an upper bound, we made the critical computations with coefficients in some finite field $\mathbb{F}_p$ instead of $\mathbb{Q}$; indeed, this can only increase the value of the upper bound. This does not change the algorithm and allows us to justify it in a more pleasant way.

Algorithm.

INPUT: an integer n.

OUTPUT: a basis of a vector space V_n.

- Step 1. Consider the set of all Lyndon words of X^n and compute the set of standard bracketings of these elements.

- Step 2. Apply the linear map $f \mapsto f_*$, defined by

$$f_* = \Pi_Y(f) + \frac{(-1)^{n-1}}{n}(f|x_0^{n-1}x_1)$$

  to the elements obtained in Step 1 and expand the result in the basis of words in Y^*.

- Step 3. Compute the kernel of the operator $\partial_{y_1}^*$ defined by

$$(\partial_{y_1}^*(\psi)|w) = (\psi|y_1 * w), \quad \text{for all } w \in Y^*,$$

  on the elements obtained in Step 2.

- Step 4. The algorithm finishes and outputs the kernel.

Remark 5.1. To compute an upper bound of $\mathrm{Dim}(\mathfrak{dmr}_0)$, we have to take into account the relations of double shuffle and regularization. In our algorithm, we introduced these relations in a precise order.

- We first consider the relations coming from the usual shuffle. Let us remind that Equation (3.9) just means that ψ is a Lie polynomial. So we can start from a basis of the free Lie algebra on X, for instance the Lyndon basis built with the bracketed Lyndon words (see [Reu93] for the definitions). It is Step 1 of our algorithm.
- Step 2 takes into account the relation coming from the regularization process.
- About Step 3, let us first remark that the kernel of $\partial^*_{y_1}$ contains the set of all elements satisfying Equation (3.10) and whose y_1 coefficient is 0. So the vector space V_n necessarily contains the weight n part of $\mathfrak{dmr}_0$ and its dimension gives the required upper bound.

Remark 5.2. All the computations of the kernels were done by a Gaussian pivot, the last pivots being completely parallel, since the memory of the computers were much smaller than the size of the considered matrices.

5.2 Brackets of τ's: lower bound

In the last subsection, we only obtained an upper bound and not the dimension of $\mathfrak{dmr}_0$ in weight n. For each odd n, let us define τ_n as an element of V_n with non-zero coefficient in y_n. We checked by computation on words of small length that the τ_k are free for the Ihara bracket up to weight 19.

If one assumes that the τ_k with $k < n$ indeed are elements of $\mathfrak{dmr}_0$, since σ_n is an indecomposable element of $\mathfrak{dmr}_0$ of weight n, the Lyndon brackets of these τ_k form, together with σ_n, a basis of the weight n part of $\mathfrak{dmr}_0$. This gives us the required lower bound for the dimension of the weight n part of $\mathfrak{dmr}_0$. In particular, this proves that τ_n belongs to $\mathfrak{dmr}_0$.

All these remarks prove that, up to weight 19, $\mathfrak{dmr}_0$ is freely generated by one element in each odd degree greater than or equal to 3. It also proves that, up to the same order, $\mathfrak{dmr}_0$ is included in $\mathfrak{grt}_1$.

6 Experimental results

The rank computations were performed on various computers. On a Pentium III 800, the computation for $d = 18$ took less than 4 hours (see Table 1).

The computation for weight 19 was done on an UltraSparc with 4 processors, lent by the university of Alberta, and took about the same time, thanks to a parallelized algorithm. The corresponding memory usage was 700 MB.

The computation for weight 20 should run in less than a day. Unfortunately, some memory problems arise: the matrix is larger than 4 GB. We could fix this by changing our implementation. Switching to a 64-bit system would be a quick, yet unsatisfying solution.

Table 1. Dimension and execution time of our algorithm on a Pentium III

Weight	Dimension of $\mathfrak{dmr}_0$	Time
5	1	
6	0	
7	1	
8	1	
9	1	
10	1	
11	2	
12	2	
13	3	
14	3	
15	4	0h04
16	5	0h15
17	7	1h
18	8	3h30
19	11	

7 Conclusion

In this paper, we proved that up to weight 19, $\mathfrak{dmr}_0$ is freely generated by one element in each odd degree greater than or equal to 3. We also obtained the inclusion $\mathfrak{dmr}_0 \subset \mathfrak{grt}_1$ which in turn means that the system of relations between MZVs coming from the associators is implied by the regularized double-shuffle relations.

We also found a surprising experimental result:

Conjecture 7.1. *The set* $\mathfrak{dmr}$ *exactly is the set of Lie polynomials* ψ *such that* $\partial_{y_1}^*(\psi) = 0$.

Let us finally mention, as perspectives, that similar computations can be done about the Lie algebra $\mathfrak{dmr}\mathfrak{d}_0(\mu_n)$ associated with the more general case of values of multiple polylogarithms at the n th roots of unity.

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Algorithmic computation of exponents for linear differential systems

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Abstract. We study the algorithmic aspects of the algebraic formal theory of exponents for systems of linear differential equations. In particular, we specify the number of coefficients in a series expansion which are needed to explicitly compute such invariants as exponents and exponential parts.

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1 Introduction

Exponents are complex numbers which measure the growth of solutions at a singular point of either a linear differential equation or a differential system. For a differential polynomial

$$L = \left(\frac{d}{dz}\right)^n + a_{n-1}(z)\left(\frac{d}{dz}\right)^{n-1} + \cdots + a_0(z) \in \mathbb{C}(z)\left[\frac{d}{dz}\right]$$

the definition of exponents at a regular singularity (*i.e.* where solutions of the differential equation $Ly = 0$ have at most polynomial growth) goes back to the very first study of these equations by Fuchs and Frobenius (1866). These exponents are easy to compute. Assuming the regular singularity to be located at $z = 0$, one can rewrite the operator L as

$$L = \theta^n + b_{n-1}(z)\theta^{n-1} + \cdots + b_0(z), \quad \text{where } \theta = z\frac{d}{dz} \text{ and } b_i \in \mathbb{C}\{z\}.$$

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The exponents $e_1, \ldots, e_n$ are then the roots of the *indicial polynomial*

$$P_{0,L}(\rho) = \rho^n + b_{n-1}(0)\rho^{n-1} + \cdots + b_0(0).$$

A differential system of order n

$$\frac{dX}{dz} = AX, \tag{1.1}$$

meromorphic over $\mathbb{P}^1(\mathbb{C})$, can be reduced to a differential equation of the previous type, by means of the "cyclic vector lemma" (*cf.* [De]). Since the non-integer part of the exponents is a meromorphic invariant of the attached D-module, one can give a definition of exponents *modulo* $\mathbb{Z}$ for regular systems. A proper notion of exponents, with a fixed integer part, at a regular singularity s has been defined by A. H. M. Levelt ([Le1]) as the maximal orders of growth $e_1^s, \ldots, e_n^s$ of a full set of solutions of (1.1) with respect to a suitable valuation. These exponents are usually impossible to compute without a previous complete computation of the solutions, except for the case of a matrix A with a simple pole at s. In this case, the exponents are the eigenvalues of the residue matrix A_{-1} of A (as studied by F. R. Gantmacher in [G]).

A differential equation at an irregular singularity is a tower of extensions of twisted regular equations. D. Bertrand et G. Laumon ([B], also see [B-L]) have shown that the factorization of L as a product of regular differential polynomials twisted by a ramified ordinary polynomial of $\frac{1}{z}$ (as given by P. Robba in [Ro]) can be used to give a consistent definition of exponents in the case of an irregular singularity, however less manageable than for regular singularities since it requires the previous computation of the ramified polynomials known as *determinant factors*.

In [Cor2], we have in turn extended Levelt's definition above to irregular singularities of *systems*. In this article, we tackle the problem of explicitly computing the exponents. The reduction algorithms for systems known so far ([H-W], [Ba], …) do not take exponents into account, and would not without a careful examination allow to compute them as a by-product. Indeed, as we showed in [Cor] and [Cor2], exponents are algebraic invariants attached to some normal forms which are "closest" to the original system. In this paper, we explicitly compute this normal form (and therefore the exponents) for a differential system whose determinant factors are not ramified. This can always be assumed after ramifying at an *a priori* known order (*cf.* [Le2]), but in practice, it remains a limitation.

In Section 2 of this paper, we briefly recall the definitions of the more convenient framework of linear *connections* over vector spaces. In Section 3, we recall the results on which the characterization of exponents rely, and refer to [Cor2] for the proofs. In Section 4, we find how many terms of the series expansion of the matrix A are needed to compute the exponents, and in Section 5 we give the explicit computer algebra procedures for computing these invariants.

2 Differential systems and connections

Let $K = \mathbb{C}((z))$ denote the field of formal meromorphic power series, and $\mathcal{O} = \mathbb{C}[[z]]$ the valuation ring of K for its z-adic valuation v. Let further Ω denote the K-vector space of differential 1-forms over $\mathbb{C}$ and $\mathrm{Der}_{\mathbb{C}}(K)$ the K-vector space of $\mathbb{C}$-derivations of K. The space $\mathrm{Der}_{\mathbb{C}}(K)$ is the K-dual of Ω.

2.1 Connections

Let V be a K-vector space of finite dimension n. A linear connection on V is an additive map

$$\nabla : V \longrightarrow V \otimes_K \Omega$$

satisfying the Leibniz rule $\nabla(fv) = v \otimes df + f\nabla v$ for all $f \in K$ and all $v \in V$. For any derivation $\tau \in \mathrm{Der}_{\mathbb{C}}(K)$ one defines a map $\nabla_\tau : V \longrightarrow V$ by composing ∇ with

$$\begin{array}{ccc} V \otimes_K \Omega & \longrightarrow & V \\ v \otimes \omega & \longmapsto & \langle \omega, \tau \rangle v = \omega(\tau)v. \end{array}$$

The additive map ∇_τ satisfies $\nabla_\tau(fv) = \tau(f)v + f\nabla_\tau(v)$ for all $f \in K$ and all $v \in V$. It is a *differential operator* on V. We will mainly work with the derivations d/dz and $\theta = zd/dz$.

A vector $v \in V$ is said to be *horizontal* for the connection ∇ if $\nabla v = 0$, which amounts to asking $\nabla_\tau(v) = 0$ for any (or equivalently every) derivation $\tau \in \mathrm{Der}_{\mathbb{C}}(K)$.

Let $(e) = (e_1, \ldots, e_n)$ denote any basis of V. We define the *matrix* $\mathrm{Mat}(\nabla_\tau, (e))$ *of the differential operator* ∇_τ *in the basis* (e) as the matrix $A = (A_{ij}) \in \mathrm{M}_n(K)$ such that

$$\nabla_\tau(e_j) = -\sum_{i=1}^{n} A_{ij} e_i \quad \text{for all } j = 1, \ldots, n.$$

We will define the valuation $v(A)$ of any matrix $A \in \mathrm{M}_n(K)$ as

$$v(A) = \min_{1 \leqslant i,j \leqslant n} v(A_{ij}).$$

Let $X = {}^t(x_1, \ldots, x_n)$ be the matrix of components of $v \in V$ in the basis (e). The matrix of components of $\nabla_\tau(v)$ in (e) is then $\tau X - AX$. The differential system $\tau X = AX$ and the equation $\nabla_\tau(v) = 0$ are therefore equivalent via the choice of a basis.

Let (ε) be a basis of V. Let $P_{(e),(\varepsilon)}$ denote the matrix $P = \mathrm{Mat}(\mathrm{id}_V, (\varepsilon), (e)) \in \mathrm{GL}_n(K)$ of the basis change from (e) to (ε). The components of v in (ε) are given by Y with $X = PY$, and the components of $\nabla_\tau(v)$ by $\tau Y - A_{[P]}Y$, where the matrix $A_{[P]}$ is the *gauge transform* of A with respect to the derivation τ

$$A_{[P]} = P^{-1}AP - P^{-1}\tau P. \tag{2.1}$$

For two bases (e) and (ε) we call $P_{(e),(\varepsilon)}$ the *associated gauge matrix*.

On the K-dual V^* of V there is a natural connection ∇^* called the *dual connection* of ∇, induced by ∇ and given for any $\tau \in \mathrm{Der}_{\mathbb{C}}(K)$ by

$$\nabla^*_\tau(f)(v) = \tau(f(v)) - f(\nabla_\tau(v)) \quad \text{for any } f \in V^*, \text{ and any } v \in V. \tag{2.2}$$

Let (e) be a basis of V and $A = \mathrm{Mat}(\nabla_\tau, (e))$ be the matrix of ∇_τ in (e). The matrix of ∇^*_τ in the dual basis (e^*) is

$$\mathrm{Mat}(\nabla^*_\tau, (e^*)) = -{}^tA.$$

Similarly, on the space $\mathrm{End}_K(V)$ of linear maps of V, there is a canonical connection $\mathrm{End}(\nabla)$ defined by

$$\mathrm{End}(\nabla)_\tau(f) = \nabla_\tau \circ f - f \circ \nabla_\tau \quad \text{for any } f \in \mathrm{End}_K(V). \tag{2.3}$$

For any endomorphism f on a vector space over a field F, let $\mathfrak{Sp}(f)$ denote the set of eigenvalues of f in F. We will use the same notation for matrices as well.

2.2 Lattices

A *lattice* Λ in a K-vector space V is a free $\mathcal{O}$-module of rank n in V, and a *sublattice* of Λ a lattice M in V such that $M \subset \Lambda$. Let $\mathcal{L}(e)$ denote the lattice spanned by a basis (e), and $\mathfrak{L}$ the set of all lattices in V.

For any lattice Λ in V, the map v_Λ defined by

$$v_\Lambda(x) = \sup\{k \in \mathbb{Z} \mid x \in z^k\Lambda\} \quad \text{for any } x \in V$$

is a valuation on V. For any lattice M in V, let

$$v_\Lambda(M) = \inf_{x \in M} v_\Lambda(x),$$

with $v_\Lambda(M) = \infty$ if $M = (0)$. This definition holds in fact for any non-empty subset M of V, provided that M has *finite depth*, *i.e.* it is contained in a finitely generated $\mathcal{O}$-module of V.

The following lemma summarizes basic properties of lattices and valuations (*cf.* [Cor2]).

Lemma 2.1. *Let $\Lambda = \mathcal{L}(e)$ and $M = \mathcal{L}(\varepsilon)$ be two lattices in V. Let P denote the gauge matrix $P_{(e),(\varepsilon)}$. Then the following hold.*

i) $v_\Lambda(M) = \min_{i=1,\dots,n} v_\Lambda(\varepsilon_i) = v(P)$.

ii) *$v_\Lambda(M)$ is the unique integer k such that $z^{-k}M \subset \Lambda$ and $z^{-k-1}M \not\subset \Lambda$. In particular, one has $v_\Lambda(z^{-v_\Lambda(M)}M) = 0$.*

iii) *For any integer $k > 0$, the lattice z^kM is a strict sublattice of M.*

iv) *$M \subset \Lambda$ if and only if $P \in \mathrm{M}_n(\mathcal{O})$ or, equivalently, if $v(P) \geqslant 0$.*

v) *$M = \Lambda$ if and only if $P \in \mathrm{GL}_n(\mathcal{O})$, or equivalently, if $P = P_0 + P_1 z + \cdots$ with $P_0 \in \mathrm{GL}_n(\mathbb{C})$.*

vi) *Let $\tilde{M}$ be a lattice in V. Then we have $v_\Lambda(M + \tilde{M}) = \min(v_\Lambda(M), v_\Lambda(\tilde{M}))$.*

Lemma 2.2. *Let N, M and Λ be three lattices in V. If $N \subset M \subset \Lambda$ holds, then $v_\Lambda(M) + v_M(N) \leqslant v_\Lambda(N)$ and $v_M(\Lambda) + v_N(M) \leqslant v_N(\Lambda)$.*

Proof. According to Lemma 2.1 ii), we have $z^{-v_\Lambda(M)} M \subset \Lambda$ and $z^{-v_M(N)} N \subset M$. Thus $z^{-v_\Lambda(M)} z^{-v_M(N)} N \subset \Lambda$. Since $v_\Lambda(N)$ is equal to the largest $k \in \mathbb{Z}$ such that $z^{-k} N \subset \Lambda$, we get $v_\Lambda(M) + v_M(N) \leqslant v_\Lambda(N)$. A similar argument yields the second statement. $\square$

Let us recall that there is a bijective correspondence between lattices in V and lattices in the dual V^* of V, given by the duality map

$$\Lambda \longmapsto \Lambda^* = \mathrm{Hom}_K(\Lambda, \mathcal{O}).$$

For any two lattices Λ and M of V, there exists a unique increasing sequence of integers $k_1 \leqslant \cdots \leqslant k_n$ and an $\mathcal{O}$-basis $(e_1, \ldots, e_n)$ of Λ such that $(z^{k_1} e_1, \ldots, z^{k_n} e_n)$ is a basis of M. We call these integers k_i *elementary divisors of M in Λ* and *Smith basis of Λ for M* any such basis. We say that a basis (ε) of M is Λ-*Smith* if $(z^{-k_1} \varepsilon_1, \ldots, z^{-k_n} \varepsilon_n)$ is a basis of Λ.

We denote the elementary divisors of M in Λ by $k_{i,\Lambda}(M)$ to specify the respective lattices. Let

$$\mathcal{K}_\Lambda(M) \;=\; (k_{1,\Lambda}(M), \ldots, k_{n,\Lambda}(M))$$

and let $z^{\mathcal{K}_\Lambda(M)}$ denote the diagonal matrix $\mathrm{diag}(z^{k_{1,\Lambda}(M)}, \ldots, z^{k_{n,\Lambda}(M)})$.

Lemma 2.3. *Let Λ be a lattice in V. For any $1 \leqslant i \leqslant n$, the map*

$$\begin{array}{rcl} k_{i,\Lambda}(\,\cdot\,)\ \ (\textit{resp. } k_{i,\cdot}(\Lambda)) : \mathfrak{L} & \longrightarrow & \mathbb{Z} \\ M & \longmapsto & k_{i,\Lambda}(M)\ \ (\textit{resp. } k_{i,M}(\Lambda)) \end{array}$$

is a decreasing (resp. increasing) map, for the ordering of $\mathfrak{L}$ by inclusion.

Proof. Consider two lattices $M \subset N$. Fix a Smith basis (e) of Λ for M and a Smith basis (ε) of Λ for N. The gauge $\tilde{P} = z^{-\mathcal{K}_\Lambda(N)} P_{(\varepsilon),(e)} z^{\mathcal{K}_\Lambda(M)}$ is a gauge from N to M, hence $\tilde{P} \in \mathrm{M}_n(\mathcal{O})$ and $v(\tilde{P}) = \min_{i,j} v(P_{ij}) + k_{j,\Lambda}(N) - k_{i,\Lambda}(M) \geqslant 0$. Since $P \in \mathrm{GL}_n(\mathcal{O})$, there exists a permutation σ such that $v(P_{i,\sigma(i)}) = 0$ for $1 \leqslant i \leqslant n$, which implies $k_{\sigma(i),\Lambda}(N) - k_{i,\Lambda}(M) \leqslant 0$. Since both sequences of elementary divisors increase, we get $k_{i,\Lambda}(N) \leqslant k_{i,\Lambda}(M)$. The other result follows from the fact that $k_{i,M}(\Lambda) = -k_{n-i,\Lambda}(M)$. $\square$

Since $k_{1,\Lambda}(M) = v_\Lambda(M)$ we get the following result.

Corollary 2.4. *Let $N \subset M \subset \Lambda$ be lattices in V. Then we have*

$$v_N(\Lambda) \leqslant \min(v_N(M), v_M(\Lambda)) \textit{ and } \max(v_M(N), v_\Lambda(M)) \leqslant v_\Lambda(N).$$

For any sublattice $M \subset \Lambda$ of a lattice Λ, the quotient module Λ/M is a finite dimensional $\mathbb{C}$-vector space.

Lemma 2.5. *Let Λ be a lattice in V. Let $\overline{\Lambda}$ denote the quotient space $\Lambda/z\Lambda$, and $\overline{x}$ the class of $x \in \Lambda$ in $\overline{\Lambda}$.*

i) *$(e_1, \dots, e_n) \in \Lambda^n$ is an $\mathcal{O}$-basis of Λ if and only if $(\overline{e}_1, \dots, \overline{e}_n)$ is a $\mathbb{C}$-basis of $\overline{\Lambda}$.*

ii) *For $m \leqslant n$, the sequence $(e_1, \dots, e_m) \in \Lambda^m$ can be completed as a basis of Λ if and only if $(\overline{e}_1, \dots, \overline{e}_m)$ is free in $\overline{\Lambda}$.*

Proof. We only prove the "if" statements.

i) Let $(\varepsilon_1, \dots, \varepsilon_n)$ be a basis of Λ. Let P be the matrix of coordinates of (e) in (ε). According to the assumption, we have $P = P_0 + \cdots$ where P_0 is the basis change from $(\overline{e}_1, \dots, \overline{e}_n)$ to $(\overline{\varepsilon}_1, \dots, \overline{\varepsilon}_n)$. Accordingly, we get $P_0 \in \mathrm{GL}_n(\mathbb{C})$. Hence $P \in \mathrm{GL}_n(\mathcal{O})$ holds, and (e) is a basis of Λ.

ii) Since $\overline{e}_1, \dots, \overline{e}_m$ are linearly independent, we can find $n-m$ vectors $v_{m+1}, \dots, v_n$ such that $(\overline{e}_1, \dots, \overline{e}_m, \overline{v}_{m+1}, \dots, \overline{v}_n)$ is a basis of $\overline{\Lambda}$. According to what has just been proved, for any such sequence $(v_{m+1}, \dots, v_n)$ the family $(e_1, \dots, e_m, v_{m+1}, \dots, v_n)$ is a basis of Λ. □

Definition 2.6. Let $M \subset \Lambda$ be two lattices in V. We define the two following quantities.

i) The *index* $[\Lambda : M]$ of M in Λ, equal to the dimension $[\Lambda : M] = \dim_{\mathbb{C}} \Lambda/M$ of the quotient module Λ/M.

ii) The *relative span* of Λ and M, given by $\delta(\Lambda, M) = -v_\Lambda(M) - v_M(\Lambda) \geqslant 0$.

Lemma 2.7. *Let $M \subset \Lambda$ be lattices in V. Let P be any gauge matrix from Λ to M. Then the following hold.*

i) $[\Lambda : M] = v(\det P) = \sum_{i=1}^n k_{i,\Lambda}(M)$.

ii) $\delta(\Lambda, M) = k_{n,\Lambda}(M) - k_{1,\Lambda}(M) = -v(P^{-1}) - v(P)$.

iii) *There exist $Q, R \in \mathrm{GL}_n(\mathcal{O})$ such that $P = Qz^{\mathcal{K}_\Lambda(M)}R$.*

iv) *Let $N \subset M$ be a lattice such that $v_\Lambda(N) = 0$. Then*

$$\max(\delta(N, M), \delta(M, \Lambda)) \leqslant \delta(N, \Lambda) \leqslant \delta(N, M) + \delta(M, \Lambda).$$

Proof. The three first assertions are either immediate or well-known, whereas iv) is a direct consequence of Lemma 2.2 and Corollary 2.4. □

Let $\Psi : V \longrightarrow V$ be either an $\mathcal{O}$-linear map or a differential operator. The *Poincaré rank of* Ψ *on the lattice* Λ is defined as the integer

$$\mathfrak{p}_\Lambda(\Psi) = -v_\Lambda(\Lambda + \Psi(\Lambda)).$$

Let us recall that the connection ∇ is *regular* if there exists a lattice in V which is stable by ∇_θ, and *irregular* otherwise. Note that, if a lattice M is ∇_θ-stable, so are all its translated lattices $z^k M$.

Definition 2.8. Let Λ be a lattice in V and let $\mathfrak{p} = \mathfrak{p}_\Lambda(\nabla)$ be the Poincaré rank of ∇ on Λ. The *polar map of* ∇ *on the lattice* Λ is the map $\overline{\nabla}^\Lambda$ induced on $\Lambda/z\Lambda$ by the operator $z^{\mathfrak{p}}\nabla_\theta$. If Λ is ∇_θ-stable, we call $\overline{\nabla}^\Lambda$ the *residue of* ∇ *on the lattice* Λ and denote it by $\operatorname{Res}_\Lambda \nabla$.

Proposition 2.9. *Let* Λ *and* M *be lattices in* V *such that* $\mathfrak{p}_\Lambda(\nabla) = \mathfrak{p}_M(\nabla)$. *Then the polar maps* $\overline{\nabla}^\Lambda$ *and* $\overline{\nabla}^M$ *have the same characteristic polynomial.*

Proof. Let $k_1, \dots, k_m$ denote the *distinct* elementary divisors of M in Λ and $n_1, \dots, n_m$ their respective multiplicities. Let $\mathcal{K}$ denote the diagonal matrix

$$\mathcal{K} = \operatorname{diag}(k_1 I_{n_1}, \dots, k_m I_{n_m}).$$

Let (e) be a Smith basis of Λ with respect to M, and (ε) the basis $z^{\mathcal{K}}(e)$ of M. Let further $A = \operatorname{Mat}(\nabla_\theta, (e)) = A_{-\mathfrak{p}} z^{-\mathfrak{p}} + \cdots$ and $B = \operatorname{Mat}(\nabla_\theta, (\varepsilon)) = B_{-\mathfrak{p}} z^{-\mathfrak{p}} + \cdots$ be the respective matrices of ∇_θ in the given bases. We have $B = z^{-\mathcal{K}} A z^{\mathcal{K}} - \mathcal{K}$, *i.e.* element-wise

$$B_{ij} = A_{ij} z^{k_j - k_i} - \delta_{ij} k_i.$$

By assumption, we have $v(B_{ij}) \geqslant -\mathfrak{p}$ hence if $k_j - k_i < 0$ implies $v(A_{ij}) > -\mathfrak{p}$. Thus, the matrix $A_{-\mathfrak{p}}$ has a block upper-triangular structure matching the blocks of the matrix $\mathcal{K}$. Similarly, the matrix $B_{-\mathfrak{p}}$ has block lower-triangular structure according to the same blocks, and moreover their diagonal blocks $A_{ii}^{(-\mathfrak{p})}$ and $B_{ii}^{(-\mathfrak{p})}$ are equal. Accordingly, we get

$$A_{-\mathfrak{p}} = \begin{pmatrix} A_{11}^{(-\mathfrak{p})} & & A_{ij}^{(-\mathfrak{p})} \\ & \ddots & \\ 0 & & A_{mm}^{(-\mathfrak{p})} \end{pmatrix} \quad \text{and} \quad B_{-\mathfrak{p}} = \begin{pmatrix} A_{11}^{(-\mathfrak{p})} & & 0 \\ & \ddots & \\ B_{ij}^{(-\mathfrak{p})} & & A_{mm}^{(-\mathfrak{p})} \end{pmatrix}.$$

The two matrices $A_{-\mathfrak{p}}$ and $B_{-\mathfrak{p}}$ have clearly the same characteristic polynomial, which is the characteristic polynomial of the respective polar maps. $\square$

2.3 Subconnections

If W is a K-subspace of V such that $\nabla(W) \subset W \otimes_K \Omega$, the restriction $\nabla_{|W}$ of ∇ to W is a connection on W. We will abusively say that W is a *subconnection* of V, or a ∇-*subconnection* of V if we need to specify the connection.

Lemma 2.10. *Let V_1, V_2 be two K-subspaces of V such that $V_1 \oplus V_2 = V$. Let Λ be a lattice such that $\Lambda = \Lambda \cap V_1 \oplus \Lambda \cap V_2$.*

i) *Let W be a K-subspace of V such that $W = (W \cap V_1) \oplus (W \cap V_2)$. Then we have*
$$\Lambda \cap W = (\Lambda \cap W) \cap V_1 \oplus (\Lambda \cap W) \cap V_2.$$

ii) *Let W_1, W_2 be two K-subspaces of V such that $W_1 \oplus W_2 = V$ and $\Lambda = \Lambda \cap W_1 \oplus \Lambda \cap W_2$. Assume moreover that $\Lambda \cap V_i / z(\Lambda \cap V_i) = \Lambda \cap W_i / z(\Lambda \cap W_i)$ for $i = 1, 2$. Then we have*
$$W_1 = V_1 \text{ and } W_2 = V_2.$$

Proof. The inclusion $\Lambda \cap W \supset (\Lambda \cap W) \cap V_1 \oplus (\Lambda \cap W) \cap V_2$ is clear. Let us consider $x \in \Lambda \cap W$. By assumption, there exist unique $x_1 \in \Lambda \cap V_1$ and $x_2 \in \Lambda \cap V_2$ and unique $y_1 \in W \cap V_1$ and $y_2 \in W \cap V_2$ such that $x = x_1 + x_2 = y_1 + y_2$. Accordingly we get $x_1 - y_1 = y_2 - x_2$. Since $V_1 \cap V_2 = (0)$, we get $x_1 = y_1 \in \Lambda \cap W \cap V_1$ and $x_2 = y_2 \in \Lambda \cap W \cap V_2$. Therefore, we find that $x \in (\Lambda \cap W) \cap V_1 \oplus (\Lambda \cap W) \cap V_2$, hence i). For $i = 1, 2$, let (e_i) (resp. (ε_i)) be bases of $V_i \cap \Lambda$ (resp. $W_i \cap \Lambda$) such that $(\overline{\varepsilon}_i) = (\overline{e}_i)$. Since $(\overline{e}_i)$ lifts either to (e_i) or (ε_i) and $(\overline{e}_1, \overline{e}_2)$ is a basis of $\Lambda / z\Lambda$, then, according to Lemma 2.5, the families $(u) = (e_1, \varepsilon_2)$ and $(v) = (\varepsilon_1, e_2)$ are bases of Λ. The attached gauge matrix satisfies then $P_{(u),(v)} = I + \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}$ with $v(Q_i) \geqslant 1$, hence (ε_1) spans V_1 and (e_2) spans W_2. $\square$

Definition 2.11. We say that a collection $\mathcal{V} = (V_1, \dots, V_s)$ of nontrivial subspaces of V is a *direct sum of* V if we have $V = V_1 \oplus \cdots \oplus V_s$. Let $\mathcal{V} = (V_1, \dots, V_s)$ and $\mathcal{W} = (W_1, \dots, W_t)$ be two direct sums of V.

i) If each V_i is a subconnection, we say that $\mathcal{V}$ is a *direct sum of subconnections* of V.

ii) We say that $\mathcal{V}$ is *finer* than $\mathcal{W}$, and we write $\mathcal{V} \prec \mathcal{W}$, if for any $1 \leqslant i \leqslant s$ there exists $1 \leqslant j \leqslant t$ such that $V_i \subset W_j$.

iii) We say that $\mathcal{V}$ and $\mathcal{W}$ are *compatible* if there exists a direct sum $\mathcal{Z}$, which is finer than $\mathcal{V}$ and $\mathcal{W}$.

iv) If $\mathcal{V}$ and $\mathcal{W}$ are compatible, by the *intersection* of $\mathcal{V}$ and $\mathcal{W}$ we mean the largest direct sum of V such that $\mathcal{V} \wedge \mathcal{W} \prec \mathcal{V}$ and $\mathcal{V} \wedge \mathcal{W} \prec \mathcal{W}$. It is the direct sum V obtained by taking all distinct non-zero intersections of V_i and W_j.

v) We write $\Lambda_{\mathcal{V}} = \bigoplus_{i=1}^{s} \Lambda \cap V_i$ for any lattice Λ, and we say that Λ is *adapted to* $\mathcal{V}$ if $\Lambda = \Lambda_{\mathcal{V}}$.

Lemma 2.12. *Let $\mathcal{V} = (V_1, \dots, V_s)$ be a direct sum of V and Λ a lattice of V.*

i) *The lattice* $\Lambda_{\mathcal{V}}$ *is the largest sublattice of* Λ *which is adapted to* $\mathcal{V}$.

ii) $\mathcal{V}^* = (V_1^*, \dots, V_s^*)$ *is a direct sum of* V^* *and* Λ *is adapted to* $\mathcal{V}$ *if and only if* Λ^* *is adapted to* $\mathcal{V}^*$.

iii) *Let* $\mathcal{V} \prec \mathcal{W}$ *be a direct sum of* V. *If* $\Lambda_{\mathcal{V}} = \Lambda$ *then* $\Lambda_{\mathcal{W}} = \Lambda$.

iv) *Let* $\mathcal{W}$ *be a direct sum of* V *which is compatible with* $\mathcal{V}$. *If* Λ *is adapted to both* $\mathcal{V}$ *and* $\mathcal{W}$, *then* Λ *is adapted to* $\mathcal{V} \wedge \mathcal{W}$.

Proof. The lattice $\bigoplus_{i=1}^s (\Lambda \cap V_i)$ is adapted to the direct sum $\bigoplus_{i=1}^s V_i$ according to its construction. Let M be a lattice in V such that $M_{\mathcal{V}} = M$ and such that $\Lambda_{\mathcal{V}} \subset M \subset \Lambda$. Taking intersections with V_i we get $\Lambda \cap V_i \subset M \cap V_i \subset \Lambda \cap V_i$ for all $i = 1, \dots, n$. Thus $M \cap V_i = \Lambda \cap V_i$ and so the equality

$$M = \bigoplus_{i=1}^{s} (M \cap V_i) = \bigoplus_{i=1}^{s} (\Lambda \cap V_i) = \Lambda_{\mathcal{V}}$$

holds, hence i). The duality bijection yields easily ii). Let $\mathcal{W} = (W_1, \dots, W_t)$. Since $\mathcal{V} \prec \mathcal{W}$, then for any i there exists j_i such that $V_i \subset W_{j_i}$. If the lattice Λ satisfies $\Lambda_{\mathcal{V}} = \Lambda$ then we have $\Lambda = \bigoplus_{i=1}^s \Lambda \cap V_i \subset \bigoplus_{i=1}^s \Lambda \cap W_{j_i}$. Accordingly, we get $\bigoplus_{j=1}^t \Lambda \cap W_j \supset \bigoplus_{i=1}^s \Lambda \cap W_{j_i} = \Lambda$, hence $\Lambda_{\mathcal{W}} = \Lambda$. Assertion iv) is a consequence of a finite induction on Lemma 2.10. □

3 Review of theoretical results

3.1 The canonical decomposition of a connection

We recall the following result from [Cor2], p. 10.

Theorem 3.1. *Let* (V, ∇) *be a* K*-vector space endowed with a connection. There exists a unique regular connection* $\nabla^r : V \longrightarrow V \otimes_K \Omega$ *such that the following holds.*

i) *The map* $\varphi = \nabla_\theta - (\nabla^r)_\theta$ *of* V *is a semi-simple endomorphism of* V.

ii) *There exists* $m \in \mathbb{N}$ *such that all eigenvalues* φ_i *of* φ *are polynomials in* $z^{1/m}$ *with no constant term.*

iii) $[(\mathrm{End}(\nabla^r))_\theta(\varphi), \varphi] = 0$.

The connection ∇^r is called the *attached regular connection* and φ the *determinant map*. Let $\rho(\nabla)$ denote the smallest such $m \in \mathbb{N}$ and call it *the ramification order of the connection* ∇. The *Katz rank* of ∇ is defined as

$$\kappa(\nabla) = -\min v(\varphi_i) \in \frac{1}{m}\mathbb{Z},$$

where v denotes the extension of the z-adic valuation to $\mathbb{C}[\frac{1}{z^{1/m}}]$. We recall that the *order* of singularity of ∇ is equal to the minimal Poincaré rank $m(\nabla)$ of ∇ on all possible lattices in V. We denote with ω the K-linear map $\omega = \nabla - \nabla^r : V \longrightarrow V \otimes_K \Omega$. We call $\nabla = \nabla^r + \omega$ *the canonical decomposition of the connection* ∇ and $\nabla_\theta = \nabla_\theta^r + \varphi$ *the canonical decomposition of the operator* ∇_θ.

Lemma 3.2. *Let $\mathcal{W} = (W_1, \ldots, W_t)$ be a direct sum of subconnections of V. If $\nabla = \nabla^r + \omega$ is the canonical decomposition of the connection ∇, then $\nabla^r_{|W_i} + \omega_{|W_i}$ is the canonical decomposition of $\nabla_{|W_i}$.*

Proof. On each subspace W_i there exists a canonical decomposition $\nabla_{|W_i} = \nabla_i^r + \omega_i$. Accordingly, we get two decompositions of ∇

$$\nabla = \nabla^r + \omega = \bigoplus_{i=1}^{t} \nabla_i^r + \bigoplus_{i=1}^{t} \omega_i$$

which both clearly satisfy the assumptions of Theorem 3.1. Hence we have $\nabla^r = \bigoplus_{i=1}^t \nabla_i^r$ and $\omega = \bigoplus_{i=1}^t \omega_i$. Thus W_i is a ∇^r-subconnection and $\omega(W_i) \subset W_i \otimes_K \Omega$ for all i. $\square$

We will from now on assume that $\rho(\nabla) = 1$. Therefore, the order of singularity $m(\nabla)$ will be equal to the Katz rank $\kappa(\nabla)$, and the determinant map φ diagonalizable in V.

3.2 Normal forms and exponents

Let $V_1, \ldots, V_s$ denote the eigenspaces of φ. The direct sum $\mathcal{V} = (V_1, \ldots, V_s)$ is a direct sum of subconnections of V. Let $m_1, \ldots, m_s$ denote their dimensions. We call $V = \bigoplus_{i=1}^s V_i$ the *direct sum decomposition of V with respect to* ∇.

Definition 3.3. A lattice Λ of V is said to be *compatible with* ∇ if the following two conditions hold.

i) $\nabla_\theta^r(\Lambda) \subset \Lambda$.

ii) $\Lambda = \bigoplus_{i=1}^s \Lambda \cap V_i$.

Lemma 3.4. *Let $\mathcal{W} = (W_1, \ldots, W_t)$ be a direct sum of subconnections of V. If the lattice Λ is compatible with ∇ then the lattice $\Lambda \cap W_i$ is compatible with $\nabla_{|W_i}$ for all $i = 1, \ldots, t$.*

Compatible lattices do exist. They are akin to Babbitt and Varadarajan's canonical forms (*cf.* [B-V] p. 45, and also [Cor2], p. 11)

Proposition 3.5. *A lattice Λ is compatible with ∇ if and only if it admits a basis (e) such that*

$$\mathrm{Mat}(\nabla_\theta, (e)) = D_{-p} z^{-p} + \cdots + D_{-1} z^{-1} + N + z^N L z^{-N} \tag{3.1}$$

where

i) D_i *are diagonal matrices,*

ii) N *is a diagonal matrix of integers whose coefficients* N_i *are arranged in decreasing order,*

iii) L *is an upper triangular matrix whose diagonal coefficients* L_{ii} *satisfy* $\mathrm{Re}(L_{ii}) \in [0, 1[$,

iv) $[D_i, L] = 0$.

Proof. Assume that Λ is compatible. There is a basis (e) of Λ such that $\mathrm{Mat}(\varphi, (e)) = D_{-p}z^{-p} + \cdots + D_{-1}z^{-1} = D$ is diagonal. According to condition iii) of Theorem 3.1, the matrix $B = \mathrm{Mat}(\nabla_\theta^r, (e))$ satisfies $[B, D] = 0$, hence it has a block-diagonal structure with respect to the eigenspaces V_i whence iv). The conditions ii) and iii) stem from the fact that each block can be put in "Gantmacher normal form" (*cf.* [Cor], prop. 3.3). The converse is easily checked. □

We call this form a *BVG normal form* (i.e. Babbitt–Varadarajan–Gantmacher normal form) for the connection matrix, and the attached basis of Λ a *BVG basis*. Note that any system of order one is in BVG normal form.

Corollary 3.6. *Let* Λ *be a lattice in* V, *and let* $\mathfrak{p} = \mathfrak{p}_\Lambda(\nabla)$ *the Poincaré rank of* ∇ *on* Λ. *If* $\overline{\nabla}^\Lambda$ *is nilpotent, then* $\mathfrak{p} > \kappa(\nabla)$.

Proof. According to Proposition 2.9, if $\mathfrak{p} = \kappa(\nabla)$, then $\mathfrak{Sp}(\overline{\nabla}^\Lambda) = \mathfrak{Sp}(\overline{\nabla}^{\Lambda_L(\nabla)})$. However, according to the proposition above, the map $\overline{\nabla}^{\Lambda_L(\nabla)}$ is never nilpotent. □

If Λ is a lattice in V, there is always a maximal compatible sublattice of Λ. Let us recall the following definitions from [Cor2].

Definition 3.7. Let Λ be a lattice in V.

i) If Λ is compatible, we call *exponents* of ∇ over Λ the eigenvalues $e_1^\Lambda(\nabla), \ldots, e_n^\Lambda(\nabla)$ of the residue $\mathrm{Res}_\Lambda \nabla^r$ of ∇^r on the lattice Λ. The exponents are equal to the numbers $N_1 + L_{11}, \ldots, N_n + L_{nn}$ that appear in formula (3.1), for any BVG basis of Λ.

ii) The largest compatible sublattice $\Lambda_L(\nabla)$ of Λ is called the *Levelt lattice of* Λ *with respect to the connection* ∇.

iii) The *exponents* $e_i^\Lambda(\nabla)$ for any lattice Λ are defined to be the exponents $e_i^{\Lambda_L}(\nabla)$ of ∇ on its Levelt lattice $\Lambda_L(\nabla)$.

The proof of the following lemma is left to the reader.

Lemma 3.8. *Let* $\nabla = \nabla^r + \omega$ *be the canonical decomposition of a connection* ∇. *For any polynomial* $P \in \frac{1}{z}\mathbb{C}\big[\frac{1}{z}\big]$, *the map* $\nabla_P = \nabla + P\,\mathrm{id}_V \otimes \frac{dz}{z}$ *is a connection on* V *and one has*

i) $\nabla^r + (\omega + P\,\mathrm{id}_V \otimes \frac{dz}{z})$ *is the canonical decomposition of* ∇_P,

ii) $\Lambda_L(\nabla_P) = \Lambda_L(\nabla)$.

Lemma 3.9. *Let* Λ *be a lattice in* V *and* $\mathfrak{p} = \mathfrak{p}_\Lambda(\nabla)$ *the Poincaré rank of* ∇ *on* Λ. *Assume that* $\mathfrak{Sp}(\overline{\nabla}^\Lambda) = \{\lambda\}$ *with* $\lambda \neq 0$. *Let* $\varphi_1, \ldots, \varphi_s$ *be the determinant factors of* ∇.

i) $\kappa(\nabla) = \mathfrak{p}$.

ii) $v(\varphi_i - \frac{\lambda}{z^{\mathfrak{p}}}) > -\mathfrak{p}$ *for any* $1 \leqslant i \leqslant s$.

Proof. Let $P = -\frac{\lambda}{z^{\mathfrak{p}}}$. The attached eigenvalues of the connection ∇_P are equal to $\tilde{\varphi}_i = \varphi_i - \frac{\lambda}{z^{\mathfrak{p}}}$, according to Lemma 3.8. By construction, we have $\mathfrak{Sp}(\nabla_P) = \{0\}$. Hence, according to Corollary 3.6, we have $\kappa(\nabla_P) < \mathfrak{p}$. Thus $v(\varphi_i - \frac{\lambda}{z^{\mathfrak{p}}}) > -\mathfrak{p}$ for all $i = 1, \ldots, s$. Therefore, all the φ_i have leading coefficient equal to $\frac{\lambda}{z^{\mathfrak{p}}}$. Hence $\kappa(\nabla) = \mathfrak{p}$. □

3.3 Other maximal lattices

Let us recall the construction of Gérard and Levelt (in [G-Le]) of a *saturated lattice*. Let Λ be a lattice in V and $\vartheta \in \mathrm{Der}_{\mathbb{C}}(K)$ a derivation on K. The *k-th saturated lattice of* Λ *with respect to* ϑ is defined as the lattice

$$\mathcal{F}^k_{\nabla_\vartheta}(\Lambda) = \Lambda + \nabla_\vartheta(\Lambda) + \cdots + \nabla^k_\vartheta(\Lambda).$$

Theorem 3.10 (Gérard-Levelt). *If the connection* ∇ *has an order of singularity* $m \in \mathbb{N}$, *then for any lattice* Λ *of* V *and any* $\ell \geqslant m$, *the saturated lattice* $\mathcal{F}^{n-1}_{z^\ell\nabla_\theta}(\Lambda)$ *of* Λ *is* $z^\ell\nabla_\theta$*-stable.*

We will need the following lemma.

Lemma 3.11. *Let* Λ *be a lattice in* V *and* $\mathfrak{p} = \mathfrak{p}_\Lambda(\nabla)$ *the Poincaré rank of* ∇ *on* Λ. *Let* m *be the order of singularity of* ∇.

i) *For any* $k \geqslant 0$, *one has* $\mathcal{F}^k_{\nabla_\vartheta}(\Lambda) + \nabla_\vartheta\big(\mathcal{F}^k_{\nabla_\vartheta}(\Lambda)\big) = \mathcal{F}^{k+1}_{\nabla_\vartheta}(\Lambda)$.

ii) *For any* $\ell \geqslant m$, *the lattice* $\mathcal{F}^{n-1}_{z^\ell\nabla_\theta}(\Lambda)$ *is the smallest* $z^\ell\nabla_\theta$*-stable lattice in* V *containing* Λ.

iii) *For any* $\mathfrak{p} \geqslant \ell \geqslant m$, *write* $\mathcal{F}_\ell = \Lambda + z^\ell\nabla_\theta(\Lambda)$. *Then one has* $v_{\mathcal{F}_\ell}(\Lambda) = 0$ *and* $\delta(\Lambda, \mathcal{F}_\ell) = \mathfrak{p}_\Lambda(\nabla) - \ell$.

Proof. We only shall prove assertion iii). Denote with $k = v_{\mathcal{F}_\ell}(\Lambda) \geqslant 0$ the valuation of Λ in its saturated lattice. Since $z^{-k}\Lambda \subset \mathcal{F}_\ell$, one has the following inclusions

$$z^k\Lambda \subset \Lambda \subset z^k(\mathcal{F}_\ell) \subset z^k\mathcal{F}^{n-1}_{z^\ell\nabla_\theta}(\Lambda) \subset \mathcal{F}^{n-1}_{z^\ell\nabla_\theta}(\Lambda).$$

According to Lemma 2.1, if $k \neq 0$, then $\mathcal{F}^{n-1}_{z^\ell\nabla_\theta}(\Lambda)$ is *not* the smallest $z^\ell\nabla_\theta$-stable lattice in V containing Λ. According to the definition of δ, one has then

$$\delta(\Lambda, \mathcal{F}_\ell) = -v_{\mathcal{F}_\ell}(\Lambda) - v_\Lambda(\mathcal{F}_\ell) = -v_\Lambda(\Lambda + z^\ell\nabla_\theta(\Lambda)) = \mathfrak{p}_\Lambda(z^\ell\nabla) = \mathfrak{p}_\Lambda(\nabla) - \ell. \ \square$$

Proposition 3.12. *Let $m = m(\nabla)$ be the order of singularity of the connection ∇, and $\mathfrak{p} = \mathfrak{p}_\Lambda(\nabla)$ the Poincaré rank of any lattice Λ of V. Then, for any $k \geqslant m$, the lattice*

$$\Lambda_k(\nabla) = \left(\mathcal{F}^{n-1}_{z^k\nabla^*_\theta}(\Lambda^*)\right)^*$$

is the unique maximal sublattice $\Lambda_k(\nabla)$ of Λ such that $\mathfrak{p}_{\Lambda_k(\nabla)}(\nabla) \leqslant k$. Moreover, the elementary divisors of $\Lambda_k(\nabla)$ in Λ satisfy

$$k_{i,\Lambda}(\Lambda_k(\nabla)) \leqslant (i-1)(\mathfrak{p}-k) \quad \textit{for } i = 1, \ldots, n.$$

Proof. By Lemma 3.11 ii), the lattice $\mathcal{F}^{n-1}_{z^k\nabla^*_\theta}(\Lambda^*)$ is the smallest $z^k\nabla^*_\theta$-stable lattice in V^* containing Λ^*. As a straightforward consequence of the bijection between lattices in V and lattices of V^*, we get that $\Lambda_k(\nabla)$ is the largest $z^k\nabla_\theta$-stable lattice in V contained in Λ. Let (ε) be a Smith basis of Λ for $M = \Lambda_k(\nabla)$. Let $\mathcal{K} = (k_1, \ldots, k_n)$ denote the sequence $(k_{1,\Lambda}(M), \ldots, k_{n,\Lambda}(M))$, and $A = \mathrm{Mat}(\nabla_\theta, (\varepsilon))$ the matrix of the connection in the basis (ε). We then have

$$\mathrm{Mat}(\nabla_\theta, (z^{\mathcal{K}}\varepsilon)) = A_{[z^{\mathcal{K}}]} = (A_{ij}z^{k_j-k_i} - \delta_{i,j}k_i)_{1\leqslant i,j\leqslant n}.$$

The lattice M has Poincaré rank $\leqslant k$. The matrix $A_{[z^{\mathcal{K}}]}$ has all coefficients in $z^{-k}\mathcal{O}$. Therefore,

$$v(A_{ij}) - k_i + k_j \geqslant -k \quad \text{for all } 1 \leqslant i, j \leqslant n. \tag{3.2}$$

The index $[\Lambda : M] = \sum_{i=1}^n k_i$ is minimal among the indices in Λ of all sublattices of Λ having Poincaré rank $\leqslant k$. For any $T = (t_1, \ldots, t_n) \in \mathbb{Z}^n$ such that $0 \leqslant t_1 \leqslant \cdots \leqslant t_n$ and $\sum_{i=1}^n t_i < \sum_{i=1}^n k_i$, the lattice spanned by $(z^T\varepsilon)$ has strictly larger Poincaré rank than k. There exists thus a couple of indices $(i_{(T)}, j_{(T)}) \in \{1, \ldots, n\}^2$ such that

$$v(A_{i_{(T)}j_{(T)}}) - t_{i_{(T)}} + t_{j_{(T)}} < -k. \tag{3.3}$$

Let ℓ be an index such that $k_{\ell+1} \geqslant 1$. Let us show that $k_{\ell+1} - k_\ell \leqslant \mathfrak{p} - k$. Let $t_i = k_i$ for $i \leqslant \ell$ and $t_i = k_i - 1$ for $i \geqslant \ell + 1$. Then there exists a pair (i, j) and $\varepsilon = -1, 0$ or 1 such that

$$-k > v(A_{ij}) - t_i + t_j = v(A_{ij}) - k_i + k_j + \varepsilon \geqslant \varepsilon - k.$$

Hence $v(A_{ij}) = k_i - k_j$ and $i \leqslant \ell \leqslant \ell + 1 \leqslant j$, and so

$$k_{\ell+1} - k_\ell \leqslant k_j - k_i = -v(A_{ij}) \leqslant \mathfrak{p} - k.$$

Hence we get $\max_{i=1,\dots,n-1} k_{i+1} - k_i \leqslant \mathfrak{p} - k$. The upper bound of the proposition follows. $\square$

Lemma 3.13. *Let $\mathcal{V} = (V_1, \dots, V_s)$ be a direct sum of subconnections of V and Λ a lattice in V. If Λ is adapted to $\mathcal{V}$, then $\mathcal{F}^1_{\nabla_\vartheta}(\Lambda)$ is also adapted to $\mathcal{V}$ for any derivation ϑ.*

Proof. First note that the inclusion $\mathcal{F}^1_{\nabla_\vartheta}(\Lambda) \supset \mathcal{F}^1_{\nabla_\vartheta}(\Lambda)_{\mathcal{V}}$ holds. The lattice Λ is adapted to $\mathcal{V}$ hence $\Lambda = \Lambda_{\mathcal{V}}$. The V_i are stable under ∇_ϑ, therefore we have

$$\nabla_\vartheta(\bigoplus_{i=1}^{s} \Lambda \cap V_i) = \bigoplus_{i=1}^{s} \nabla_\vartheta(\Lambda \cap V_i).$$

Since $\mathcal{V}$ is a direct sum, we have $\nabla_\vartheta(\Lambda \cap V_i) = \nabla_\vartheta(\Lambda) \cap V_i$. Accordingly, we have

$$\mathcal{F}^1_{\nabla_\vartheta}(\Lambda) = \Lambda + \nabla_\vartheta \Lambda = \bigoplus_{i=1}^{s} \Lambda \cap V_i + \bigoplus_{i=1}^{s} \nabla_\vartheta(\Lambda \cap V_i) = \bigoplus_{i=1}^{s} (\Lambda \cap V_i + \nabla_\vartheta(\Lambda) \cap V_i).$$

Since $(\Lambda \cap V_i) + (\nabla_\vartheta(\Lambda) \cap V_i) \subset (\Lambda + \nabla_\vartheta(\Lambda)) \cap V_i$ holds, we get

$$\mathcal{F}^1_{\nabla_\vartheta}(\Lambda) \subset \bigoplus_{i=1}^{s} (\Lambda + \nabla_\vartheta(\Lambda)) \cap V_i = \mathcal{F}^1_{\nabla_\vartheta}(\Lambda)_{\mathcal{V}}.$$

$\square$

Definition 3.14. Let Λ be a lattice. We call *Katz lattice of ∇ attached to the lattice Λ* the largest sublattice $\Lambda_K(\nabla) = \Lambda_{\kappa(\nabla)}(\nabla)$ of Λ having minimal Poincaré rank $\kappa(\nabla)$.

Corollary 3.15. *Let $\mathcal{V} = (V_1, \dots, V_s)$ be a direct sum of ∇-subconnections of V. Consider a lattice Λ adapted to $\mathcal{V}$. Then we have*

i) $\Lambda_k(\nabla) = \bigoplus_{i=1}^{s} (\Lambda \cap V_i)_k(\nabla_{|V_i})$. *In particular $\Lambda_k(\nabla)$ is adapted to $\mathcal{V}$ for any $k \geqslant m(\nabla)$.*

ii) $\Lambda_K(\nabla)$ *is adapted to $\mathcal{V}$.*

Proof. According to Proposition 3.12, we have $\Lambda_k(\nabla) = \left(\mathcal{F}^{n-1}_{z^k \nabla^*_\theta}(\Lambda^*)\right)^*$. Lemma 2.12 ensures us that Λ^* is adapted to $\mathcal{V}^*$, and from Lemma 3.11 i) and Lemma 3.13, it follows that $\mathcal{F}^{n-1}_{z^k \nabla^*_\theta}(\Lambda^*)$ is also adapted to $\mathcal{V}^*$. Hence $\Lambda_k(\nabla)$ is adapted to $\mathcal{V}^{**} = \mathcal{V}$. Since $\Lambda_K(\nabla) = \Lambda_{\kappa(\nabla)}(\nabla)$, result ii) holds. $\square$

Proposition 3.16. *Let Λ be a lattice in V and $\mathfrak{p} = \mathfrak{p}_\Lambda(\nabla)$ the Poincaré rank of ∇ over Λ. Let Λ_K and Λ_L the attached Katz and Levelt lattices with respect to ∇. Then*

$$\Lambda_L = (\Lambda_K)_L(\nabla).$$

Proof. Λ_K is the largest sublattice of Λ having Poincaré rank $\kappa(\nabla)$. The Levelt lattice Λ_L has also Poincaré rank $\kappa(\nabla)$. Therefore, we have $\Lambda_L \subset \Lambda_K$. Thus we get $\Lambda_L = (\Lambda_L)_L(\nabla) \subset (\Lambda_K)_L(\nabla)$. The lattice $(\Lambda_K)_L(\nabla)$ is compatible with ∇. However, there is no strictly larger lattice compatible with ∇ than Λ_L. Hence $(\Lambda_K)_L(\nabla) = \Lambda_L$. $\square$

We summarize the previous results in the following scheme, where the arrows denote inclusion maps, and where $\mathcal{V} = (V_1, \dots, V_s)$ is the attached direct sum, $\nabla_i^r = \nabla^r_{|V_i}$ and $\Lambda_i = \Lambda \cap V_i$.

$$\begin{tikzcd}
& \Lambda & \\
\Lambda_{\mathcal{V}} = \bigoplus_{i=1}^{s}(\Lambda \cap V_i) \arrow[ur] & \Lambda_K(\nabla) \arrow[u] & \Lambda_L(\nabla^r) \arrow[ul] \\
& \Lambda_L(\nabla) = \bigoplus_{i=1}^{s}(\Lambda_i)_L(\nabla_i^r) \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

$$\{e_i^{\Lambda}(\nabla)\} = \mathfrak{S}\mathfrak{p}(\mathrm{Res}_{\Lambda_L(\nabla)}\nabla^r) \qquad \begin{aligned} \Lambda_L(\nabla) &= \textstyle\bigoplus_{i=1}^{s}(\Lambda_i)_L(\nabla_i^r) \\ &= \Lambda_L(\nabla^r)_{\mathcal{V}} \\ &= \Lambda_K(\nabla)_L(\nabla) \end{aligned}$$

Sibuya's lemma (*cf.* [Le2], p. 10) is an essential tool for the formal reduction algorithms at an irregular singularity.

Proposition 3.17 (Sibuya). *Let Λ be a lattice in V, let π be the canonical projection of Λ on $\overline{\Lambda} = \Lambda/z\Lambda$ and $\delta = \overline{\nabla}^{\Lambda}$ the induced polar map on the n-dimensional $\mathbb{C}$-vector space $\overline{\Lambda}$. Let*

$$\overline{\Lambda} = F_1 \oplus F_2$$

be a direct sum of δ-stable $\mathbb{C}$-subspaces of $\overline{\Lambda}$ such that $\delta_{|F_1}$ and $\delta_{|F_2}$ have no common eigenvalue. Then there exists a ***unique*** *direct sum decomposition $V = W_1 \oplus W_2$ of V into subconnections such that the free $z^{\mathfrak{p}}\nabla_\theta$-stable $\mathcal{O}$-submodules $\Lambda_1 = \Lambda \cap W_1$ and $\Lambda_2 = \Lambda \cap W_2$ of Λ satisfy*

1) $\Lambda = \Lambda_1 \oplus \Lambda_2$,

2) $F_i = \pi(\Lambda_i)$ *for* $i = 1, 2$.

We say that (W_1, W_2) is the ***Sibuya lifting*** *of the direct sum $F_1 \oplus F_2$.*

We will use the following corollary of this result.

Corollary 3.18. *With the same notation as above, let $F_1, \dots, F_t$ be the characteristic spaces of δ. Then there exists a unique lifting of $F_1 \oplus \cdots \oplus F_t$ into a direct sum of subconnections $\mathcal{W} = (W_1, \dots, W_t)$ of V such that Λ is adapted to $\mathcal{W}$. We call*

*$\mathcal{W} = (W_1, \ldots, W_t)$ the **Sibuya splitting** of ∇ according to Λ. Moreover, Λ is adapted to the direct sum $\mathcal{W}$.*

The lattice Λ admits bases which respect the direct sum $V = W_1 \oplus \cdots \oplus W_t$, which we will call *Sibuya bases* of Λ.

Proposition 3.19. *Let $\nabla_\theta = \nabla_\theta^r + \varphi$ be the canonical decomposition attached to the connection ∇. Let $V_1, \ldots, V_s$ denote the eigenspaces of the determinant map φ and $\mathfrak{Sp}(\varphi) = \{\varphi_1, \ldots, \varphi_s\}$ the distinct determinant factors of ∇. Let Λ be a lattice, and $\mathfrak{p}$ the Poincaré rank of ∇ on Λ. If the polar map $\overline{\nabla}^\Lambda$ is not nilpotent, then we have $\kappa(\nabla) = \mathfrak{p}$. Moreover, let $\mathcal{W} = (W_1, \ldots, W_t)$ be the Sibuya splitting of ∇ according to Λ. Then*

$$W_i = \bigoplus_{j \in J_i} V_j$$

where $J_i = \{1 \leqslant j \leqslant s \mid v(\varphi_j - \frac{\lambda_i}{z^{\mathfrak{p}}}) > -\mathfrak{p}\}$.

Proof. Let $\nabla_i = \nabla_{|W_i}$ and $\Lambda_i = \Lambda \cap W_i$. According to Corollary 3.18, we have $\Lambda = \bigoplus_{i=1}^s \Lambda_i$, and $\mathfrak{Sp}\big(\overline{\nabla_i}^{\Lambda_i}\big) = \{\lambda_i\}$. The subspaces W_i are stable under the map φ, hence $\mathfrak{Sp}(\varphi) = \coprod_{i=1}^t \mathfrak{Sp}(\varphi_{|W_i})$. According to Lemma 3.9, if $\lambda_i \neq 0$, then the eigenvalues φ_j of $\varphi_{|W_i}$ satisfy

$$v\left(\varphi_j - \frac{\lambda_i}{z^{\mathfrak{p}}}\right) > -\mathfrak{p}. \tag{3.4}$$

Thus $W_i = \bigoplus_{j \in J_i} V_j$.

Since $\mathfrak{p} = \max_{i=1,\ldots,t}(\mathfrak{p}_{\Lambda_i}(\nabla_i))$ holds, we get $\mathfrak{p} = \max_{i=1,\ldots,t} -v(\varphi_i) = \kappa(\nabla)$. There is at most one eigenvalue, say λ_1, which may be 0. If this happens, then one has $v(\varphi_j - \frac{\lambda_1}{z^{\mathfrak{p}}}) = v(\varphi_j) > -\mathfrak{p}$ for the last n_1 eigenvalues of φ, hence $W_1 = \bigoplus_{j \in J_1} V_j$. □

Corollary 3.20. *Let $\mathcal{V} = (V_1, \ldots, V_s)$ denote the direct sum decomposition of V with respect to ∇. Let Λ be a lattice in V, and let $\mathcal{W}$ be the Sibuya splitting of ∇ according to Λ. Then $\mathcal{V} \prec \mathcal{W}$ and the Levelt lattice Λ_L of Λ satisfies*

$$\Lambda_L = (\Lambda_L)_{\mathcal{W}}.$$

Proof. We have $\mathcal{V} \prec \mathcal{W}$ as a direct consequence of Proposition 3.19. According to the definition, Λ_L is adapted to $\mathcal{V}$. Lemma 2.12 iii) implies then that $(\Lambda_L)_{\mathcal{W}} = \Lambda_L$. □

For Section 3.4, let us point out the following result.

Corollary 3.21. *Let Λ be a lattice in V. When $\rho(\nabla) = 1$, then $\Lambda_K = \Lambda$ if and only if $\overline{\nabla}^\Lambda$ is not nilpotent.*

Lemma 3.22. *Let $\mathcal{Z} = (Z_1, \ldots, Z_m)$ be a direct sum of subconnections of V, and Λ a lattice compatible with $\mathcal{Z}$. Let $\mathcal{W} = (W_1, \ldots, W_t)$ be the Sibuya splitting of ∇ according to Λ. Then the direct sums $\mathcal{Z}$ and $\mathcal{W}$ are compatible, and Λ is adapted to the intersection $\mathcal{Z} \wedge \mathcal{W}$.*

Proof. For any $1 \leqslant i \leqslant m$, the intersection $\Lambda_i = \Lambda \cap Z_i$ is a lattice in V_i. Let $\nabla_i = \nabla_{|Z_i}$ and let $\mathcal{W}_i = (W_1^{(i)}, \ldots, W_{t_i}^{(i)})$ be the Sibuya splitting of ∇_i with respect to Λ_i. We have

$$\bigoplus_{i=1}^{m} \bigoplus_{j=1}^{t_i} W_j^{(i)} = \bigoplus_{i=1}^{m} V_i = V.$$

Hence, the collection $\tilde{\mathcal{Z}} = (W_j^{(i)})_{1 \leqslant i \leqslant m, 1 \leqslant j \leqslant t_i}$ is a direct sum of V. The direct sum $\tilde{\mathcal{Z}}$ is finer than $\mathcal{Z}$ by construction. Let us prove that it is also finer than $\mathcal{W}$. Let $E = \Lambda / z\Lambda$ and $E = F_1 \oplus \cdots \oplus F_t$ be the sum of E into the characteristic spaces of $f = \overline{\nabla}^{\Lambda}$. For any $1 \leqslant i \leqslant m$, let $\Lambda_i / z\Lambda_i = G_i$ and $\overline{\nabla_i}^{\Lambda_i} = f_i$. Let $G_i = F_1^{(i)} \oplus \cdots \oplus F_{t_i}^{(i)}$ be the direct sum of the characteristic spaces of f_i, and $\lambda_j^{(i)}$ the corresponding eigenvalue. Every $F_j^{(i)}$ is contained in exactly one F_k. Hence we have $\sum_{\lambda_j^{(i)} = \lambda_k} F_j^{(i)} \subset F_k$. The characteristic polynomial $P_f(X)$ satisfies of $P_f(X) = \prod_{i=1}^{t} P_{f_i}(X)$ hence

$$\sum_{\lambda_j^{(i)} = \lambda_k} F_j^{(i)} = F_k.$$

Write $X_k = \bigoplus_{\lambda_j^{(i)} = \lambda_k} W_j^{(i)}$. We have $\bigoplus_{i=1}^{t} X_k = \bigoplus_{i=1}^{t} W_k$ and the images of X_k and W_k in $\Lambda / z\Lambda$ are equal. Moreover, Λ is adapted to both direct sums. According to Lemma 2.5, we get $X_k = W_k$ and hence every $W_j^{(i)}$ is contained in at least one W_k, thus $\tilde{\mathcal{Z}} \prec \mathcal{W}$. Accordingly, $\mathcal{Z}$ and $\mathcal{W}$ are compatible. Furthermore, since Λ is adapted to both $\mathcal{Z}$ and $\mathcal{W}$, and it follows from Lemma 2.12 that it is adapted to $\mathcal{Z} \wedge \mathcal{W}$. $\square$

Definition 3.23. Let Λ be a lattice in V, let $\lambda_1, \ldots, \lambda_t$ be the eigenvalues of $\overline{\nabla}^{\Lambda}$ and $\mathcal{W} = (W_1, \ldots, W_t)$ be the Sibuya splitting of ∇ according to Λ.

i) We call *Sibuya map* of ∇ on Λ the endomorphism f_{Λ}^{∇} of V given by

$$f_{\Lambda}^{\nabla} = \frac{1}{z^{\mathfrak{p}}} \sum_{i=1}^{t} \lambda_i \, \mathrm{id}_{W_i}.$$

We also write $\omega_{\Lambda}^{\nabla} = f_{\Lambda}^{\nabla} \otimes \frac{dz}{z}$.

ii) We call *Sibuya reduction* of ∇ on Λ the connection ∇_{Λ} defined as

$$\nabla_{\Lambda} = \nabla - \omega_{\Lambda}^{\nabla}.$$

Lemma 3.24. *Let* $\nabla = \nabla^r + \omega$ *be the canonical decomposition of the connection* ∇. *Let* Λ *be a lattice in* V *and* $\mathfrak{p} = \mathfrak{p}_\Lambda(\nabla)$. *Let* (e) *be a Sibuya basis of* Λ, *and* $\mathrm{Mat}(\nabla_\theta, (e)) = A_{-\mathfrak{p}} z^{-\mathfrak{p}} + \cdots$ *the matrix of* ∇_θ *in* (e).

i) *The matrix of* f_Λ^∇ *in* (e) *satisfies*
$$\mathrm{Mat}(f_\Lambda^\nabla, (e)) = (A_{-\mathfrak{p}})_s z^{-\mathfrak{p}}$$
where $(A_{-\mathfrak{p}})_s$ *denotes the semi-simple part of* $A_{-\mathfrak{p}}$. *In particular,* $\mathrm{Mat}(f_\Lambda^\nabla, (e))$ *is diagonal.*

ii) $\nabla^r + \big(\omega - \omega_\Lambda^\nabla\big)$ *is the canonical decomposition of the connection* ∇_Λ.

iii) *The Levelt lattice of* Λ *satisfies* $\Lambda_L(\nabla) = \bigoplus_{i=1}^t (\Lambda \cap W_i)_L(\nabla) = \Lambda_L(\nabla_\Lambda)$.

Proof. The map f_Λ^∇ is diagonal in any Sibuya basis of Λ and the given form stems directly from its definition. For each $i = 1, \ldots, t$, the restriction ∇_i of ∇ to W_i has canonical decomposition $\nabla_i = \nabla^r_{|W_i} + \omega_{|W_i}$. The restriction of $(f_\Lambda^\nabla)_{|W_i}$ is scalar and its unique eigenvalue is an element of $\frac{1}{z}\mathbb{C}\big[\frac{1}{z}\big]$. According to Lemma 3.8, $\nabla_i = \nabla^r_{|W_i} + \big(\omega_{|W_i} - (f_\Lambda^\nabla)_{|W_i} \otimes \frac{dz}{z}\big)$ is the canonical decomposition of ∇_i, hence ii). The assertion iii) follows. $\square$

3.4 Computation of the Levelt lattice

Proposition 3.25. *Let* Λ *be a lattice in* V *such that* $\Lambda_K(\nabla) = \Lambda$. *Then one has*

i) $\Lambda_K(\nabla_\Lambda) \subsetneq \Lambda$.

ii) $\kappa(\nabla_\Lambda) < \kappa(\nabla)$.

iii) $f_\Lambda^\nabla \neq 0$.

Proof. By construction, the connection ∇_Λ has a nilpotent polar map $\overline{\nabla_\Lambda}^\Lambda$. According to Corollary 3.21, the lattice Λ cannot be equal to $\Lambda_K(\nabla_\Lambda)$, which proves i). Moreover, we have $\kappa(\nabla) = \mathfrak{p}_\Lambda(\nabla)$. Thus if we had $\kappa(\nabla_\Lambda) = \kappa(\nabla)$, then one would have $\Lambda = \Lambda_K(\nabla_\Lambda)$, a contradiction. This proves ii). The definition of f_Λ^∇ does not imply that $f_\Lambda^\nabla \neq 0$. However, since $\Lambda_K = \Lambda$ the polar map $\overline{\nabla}^\Lambda$ of ∇ over $\Lambda = \Lambda_K(\nabla)$ is not nilpotent, and hence $f_\Lambda^\nabla \neq 0$. $\square$

This proposition gives way to the following procedure, that we call algorithm, although it will only become clear that it is one after Section 5.2.

Algorithm LEVELT. Input: a lattice Λ and a connection ∇.

Step 1. Compute the Poincaré rank $p = \mathfrak{p}_\Lambda(\nabla)$. If $p = 0$ then terminate the algorithm.

Step 2. Compute the Katz lattice $M = \Lambda_K(\nabla)$.

Step 3. Compute the map $f_M(\nabla)$ and the connection ∇_M. Replace ∇ with ∇_M and Λ with M and go to step 1.

Output: the Levelt lattice $\Lambda_L(\nabla)$ and the determinant map φ.

Theorem 3.26. *The algorithm LEVELT applied to a connection ∇ and a lattice Λ terminates after a finite number $N \leqslant \mathfrak{p}_\Lambda(\nabla)$ of loops and gives the Levelt lattice of the original lattice.*

Proof. At the cost of computing first the Katz lattice $\Lambda_K(\nabla)$ of Λ, we can assume that the Sibuya map f_Λ^∇ is not zero. Let $\mathcal{W} = (W_1, \dots, W_t)$ be the Sibuya splitting of ∇ according to Λ. Let $f_0 = f_\Lambda^\nabla$ and $\nabla_1 = \nabla_\Lambda$. The direct sum $\mathcal{W}$ is a direct sum of ∇-subconnections of V. We know that $\Lambda_\mathcal{W} = \Lambda$. Hence, according to Lemma 3.15 ii), the Katz lattice $\Lambda_1 = \Lambda_K(\nabla_1)$ of Λ is also adapted to $\mathcal{W}$. Although the direct sum $\mathcal{W}$ is not necessarily the Sibuya splitting of ∇_1 on Λ_1, it is a direct sum of ∇_1-*subconnections* of V. Indeed, the map f_0 is by construction scalar on each W_i. Hence, one has

$$\nabla_1(W_i) \subset \nabla(W_i) + \Big(f_0 \otimes \frac{dz}{z}\Big)(W_i) \subset W_i \otimes_K \Omega.$$

Accordingly, we get $(\Lambda_1)_L(\nabla_1) = (\Lambda_K(\nabla_1))_L(\nabla_1) = \Lambda_L(\nabla_1) = \Lambda_L(\nabla)$. Moreover, $\nabla_1 = \nabla^r + \big(\omega - f_0 \otimes \frac{dz}{z}\big)$ is the canonical decomposition of ∇_1.

Let now $(\nabla_\ell, \mathcal{V}_\ell, \Lambda_\ell)$ be the following triple.

a_ℓ) ∇_ℓ is a connection on V having canonical decomposition $\nabla_\ell = \nabla^r + \omega_\ell$,

b_ℓ) $\mathcal{V}_\ell = (V_i^{(\ell)})$ is a direct sum of ∇_ℓ-subconnections of V,

c_ℓ) Λ_ℓ is a lattice in V adapted to $\mathcal{V}_\ell$,

such that

A_ℓ) the Sibuya map $f_\ell = f_{\Lambda_\ell}^{\nabla_\ell}$ of ∇_ℓ on Λ_ℓ is not trivial,

B_ℓ) $(\Lambda_\ell)_L(\nabla_\ell) = \Lambda_L(\nabla)$.

Let $\mathcal{W}_\ell = (W_i^{(\ell)})$ be the Sibuya splitting of ∇_ℓ according to Λ_ℓ. Let $\nabla_{\ell+1} = (\nabla_\ell)_{\Lambda_\ell}$. Let also $\Lambda_{\ell+1} = (\Lambda_\ell)_K(\nabla_\ell)$. By construction, $\mathcal{W}_\ell$ is a direct sum of ∇_ℓ-subconnections. The direct sums $\mathcal{V}_\ell$ and $\mathcal{W}_\ell$ are compatible according to Lemma 3.22, so $\mathcal{V}_{\ell+1} = \mathcal{V}_\ell \wedge \mathcal{W}_\ell$ is a direct sum of ∇_ℓ-subconnections of V. Moreover $\mathcal{V}_{\ell+1}$ is a direct sum of $\nabla_{\ell+1}$-subconnections of V. Indeed, f_ℓ is scalar on each $W_i^{(\ell)}$, and since $\mathcal{V}_{\ell+1} \prec \mathcal{W}_\ell$, we have $f_\ell(V_i^{(\ell+1)}) \subset V_i^{(\ell+1)}$. On the other hand, since $\mathcal{V}_\ell$ and $\mathcal{W}_\ell$ are direct sums of ∇_ℓ-subconnections, we get

$$\nabla_{\ell+1}(V_i^{(\ell+1)}) \subset \nabla_\ell(V_i^{(\ell+1)}) + \Big(f_\ell \otimes \frac{dz}{z}\Big)(V_i^{(\ell+1)}) \subset V_i^{(\ell+1)} \otimes_K \Omega.$$

Define $\Lambda_{\ell+1}$ as the Katz lattice $\Lambda_{\ell+1} = (\Lambda_\ell)_K(\nabla_\ell)$. The lattice $\Lambda_{\ell+1}$ has by Corollary 3.21 a non-nilpotent polar map, hence the map $f_{\ell+1} = f_{\Lambda_{\ell+1}}^{\nabla_{\ell+1}}$ is not zero. The lattice $\Lambda_{\ell+1}$ is also adapted to both direct sums $\mathcal{V}_\ell$ and $\mathcal{W}_\ell$, hence it is adapted to $\mathcal{V}_{\ell+1}$. Furthermore, we have

$$(\Lambda_{\ell+1})_L(\nabla_{\ell+1}) = \big((\Lambda_\ell)_K(\nabla_{\ell+1})\big)_L(\nabla_{\ell+1}) = (\Lambda_\ell)_L(\nabla_{\ell+1}) = (\Lambda_\ell)_L(\nabla_\ell) = \Lambda_L(\nabla).$$

Finally, since f_ℓ is scalar on each $W_i^{(\ell)}$, we have the canonical decomposition

$$\nabla_{\ell+1} = \nabla^r + \Big(\omega_\ell - f_{\ell+1} \otimes \frac{dz}{z}\Big). \tag{3.5}$$

Hence conditions $(A_{\ell+1})$ and $(B_{\ell+1})$ are satisfied by the triple $(\nabla_{\ell+1}, \mathcal{V}_{\ell+1}, \Lambda_{\ell+1})$.

According to Lemma 3.25 ii), we have $\kappa(\nabla_{\ell+1}) < \kappa(\nabla_\ell)$. Putting $\kappa = \kappa(\nabla)$ we get $\kappa(\nabla_\kappa) = 0$. Accordingly, the connection ∇_κ is regular, thus equation (3.5) yields $\nabla_\kappa = \nabla^r$ and $\omega = \sum_{i=1}^{\kappa} f_i \otimes \frac{dz}{z}$. Thus the direct sum $\mathcal{V}_\kappa$ is the direct sum $\mathcal{V}$ of eigenspaces of φ. Since the lattice Λ_κ is adapted to the direct sums $\mathcal{V}_\ell$ for all $i = 1, \dots, \kappa$, it is adapted to $\mathcal{V}$. According to the induction assumption, we have $\Lambda_\kappa \supset \Lambda_L(\nabla)$, hence $\Lambda_\kappa = \Lambda_L(\nabla)$. □

Corollary 3.27. *Let ∇ be a connection on V and Λ be a lattice in V. Let $\mathfrak{p} = \mathfrak{p}_\Lambda(\nabla)$ be the Poincaré rank of ∇ on Λ. Let $\Lambda_L(\nabla)$ denote the Levelt lattice of Λ. Then we have*

$$k_{n,\Lambda}(\Lambda_L(\nabla)) \leqslant (n-1)\mathfrak{p}.$$

Proof. According to the previous result there is a descending chain of lattices

$$\begin{aligned}\Lambda_0 = \Lambda \supset \Lambda_1 = \Lambda_K(\nabla) \supset \Lambda_2 &\\ &= (\Lambda_1)_K(\nabla_1) \supset \cdots \supset \Lambda_\kappa\\ &= (\Lambda_{\kappa-1})_K(\nabla_{\kappa-1}) = \Lambda_L(\nabla).\end{aligned}$$

According to Lemma 2.2, we have

$$k_{n,\Lambda}(\Lambda_L(\nabla)) \leqslant k_{n,\Lambda}(\Lambda_1) + k_{n,\Lambda_1}(\Lambda_2) + \cdots + k_{n,\Lambda_{\kappa-1}}(\Lambda_\kappa).$$

Proposition 3.12 yields $k_{n,\Lambda_\ell}(\Lambda_K(\nabla_\ell)) \leqslant (n-1)(\kappa_\ell - \kappa_{\ell+1})$ where κ_ℓ stands for the Katz rank of the connection ∇_ℓ. Hence we get

$$\begin{aligned}k_{n,\Lambda}(\Lambda_L(\nabla)) &\leqslant (n-1)(\mathfrak{p}-\kappa) + (n-1)(\kappa-\kappa_1) + \cdots + (n-1)\kappa_\kappa\\ &\leqslant (n-1)\mathfrak{p}.\end{aligned}$$

□

4 Finiteness results

4.1 Perturbation of connections

Definition 4.1. Let Λ be a lattice in V. For any integer $T \in \mathbb{Z}$, we define the Λ-*equivalence up to order* T relation $\overset{\Lambda}{\underset{T}{\sim}}$ as follows.

i) $u \overset{\Lambda}{\underset{T}{\sim}} \tilde{u}$ if $u - \tilde{u} \in z^{T+1}\Lambda$ for vectors u and $\tilde{u}$ in V.

ii) $\Psi \overset{\Lambda}{\underset{T}{\sim}} \tilde{\Psi}$ if $\Psi - \tilde{\Psi} \in z^{T+1}\Lambda \otimes_{\mathcal{O}} \Lambda^*$ for maps Ψ and $\tilde{\Psi} \in \mathrm{End}_K\, V \overset{can}{\simeq} V \otimes_K V^*$.

iii) $\nabla \overset{\Lambda}{\underset{T}{\sim}} \tilde{\nabla}$ if $\nabla_\theta - \tilde{\nabla}_\theta \in z^{T+1}\Lambda \otimes_{\mathcal{O}} \Lambda^*$ for connections ∇ and $\tilde{\nabla}$ on V.

iv) Let (ε) and $(\tilde{\varepsilon})$ be bases of V. Then $(\varepsilon) \overset{\Lambda}{\underset{T}{\sim}} (\tilde{\varepsilon})$ if $\varepsilon_i \overset{\Lambda}{\underset{T}{\sim}} \tilde{\varepsilon}_i$ for all $i = 1, \ldots, n$.

Lemma 4.2. *With the given notation, one has*

i) $(\varepsilon) \overset{\Lambda}{\underset{T}{\sim}} (\tilde{\varepsilon})$ *if and only if* $P - \tilde{P} \in z^{T+1}\mathrm{M}_n(\mathcal{O})$, *where* $P = P_{(e),(\varepsilon)}$ *and* $\tilde{P} = P_{(e),(\tilde{\varepsilon})}$ *for any basis* (e) *of* Λ.

ii) $\nabla \overset{\Lambda}{\underset{T}{\sim}} \tilde{\nabla}$ *if and only if* $A - \tilde{A} \in z^{T+1}\mathrm{M}_n(\mathcal{O})$, *where* $A = \mathrm{Mat}(\nabla_\theta, (e))$ *and* $\tilde{A} = \mathrm{Mat}(\nabla_\theta, (e))$ *for any basis* (e) *of* Λ.

Proof. By definition, P (resp. $\tilde{P}$) is the matrix of $\Psi = \sum_{i=1}^n \varepsilon_i \otimes e_i$ (resp. $\tilde{\Psi} = \sum_{i=1}^n \tilde{\varepsilon}_i \otimes e_i$) in the basis $(e \otimes e^*)$. One has $\Psi - \tilde{\Psi} \in z^{T+1}\Lambda \otimes_{\mathcal{O}} \Lambda^*$ if and only if

$$v_{\Lambda \otimes_{\mathcal{O}} \Lambda^*}\Big(\sum_{i=1}^n (\varepsilon_i - \tilde{\varepsilon}_i) \otimes e_i^*\Big) \geqslant T + 1$$

which is clearly equivalent to $P - \tilde{P} \in z^{T+1}\mathrm{M}_n(\mathcal{O})$. A similar argument holds for the second assertion of the lemma. □

Lemma 4.3. *Any two matrices* $A, B \in \mathrm{M}_n(K)$ *satisfy* $v(AB) \geqslant v(A) + v(B)$.

Proof. Indeed one has

$$v(AB) = \min_{i,j} v\Big(\sum_{k=1}^n A_{ik}B_{kj}\Big) \geqslant \min_{i,j} \min_k \big(v(A_{ik}) + v(B_{kj})\big) \geqslant v(A) + v(B).$$

□

Lemma 4.4. *Let Λ be a lattice in V, and (ε),$(\tilde{\varepsilon})$ two bases of V. Let $\mu = v_{\mathcal{L}(\varepsilon)}(\Lambda)$ denote the valuation of the lattice Λ in $\mathcal{L}(\varepsilon)$. Then*

$$(\varepsilon) \underset{-\mu}{\overset{\Lambda}{\sim}} (\tilde{\varepsilon}) \textit{ implies } \mathcal{L}(\tilde{\varepsilon}) = \mathcal{L}(\varepsilon).$$

Proof. Let (e) be any basis of Λ. Denote the respective associated matrices with $P = P_{(e),(\varepsilon)}$ and $\tilde{P} = P_{(e),(\tilde{\varepsilon})}$. From $I = P^{-1}P = P^{-1}(\tilde{P} + P - \tilde{P}) = P^{-1}\tilde{P} + P^{-1}(P - \tilde{P})$ we get

$$v(P^{-1}\tilde{P} - I) = v\big(-P^{-1}(P - \tilde{P})\big) \geqslant v(P^{-1}) + v(P - \tilde{P}) = \mu - \mu + 1 = 1$$

that is $P^{-1}\tilde{P} \in \mathrm{GL}_n(\mathcal{O})$, hence $\mathcal{L}(\tilde{\varepsilon}) = \mathcal{L}(\varepsilon)$. □

Lemma 4.5. *Let ∇ and $\tilde{\nabla}$ be connections on V, and Λ a lattice in V. Assume that $\nabla \underset{T}{\overset{\Lambda}{\sim}} \tilde{\nabla}$ with $T \in \mathbb{Z}$.*

i) *Let M be a lattice in V. Then $\nabla \underset{T'}{\overset{M}{\sim}} \tilde{\nabla}$, where $T' = T - \delta(\Lambda, M)$.*

ii) *For any $\ell \geqslant m$, let $\mathcal{F}_\ell = \Lambda + z^\ell \nabla_\theta(\Lambda)$. Then $\nabla \underset{T'}{\overset{\mathcal{F}_\ell}{\sim}} \tilde{\nabla}$ with $T' = T - \mathfrak{p}_\Lambda(\nabla) + \ell$.*

iii) *If $T \geqslant -\ell - 1$, then $\Lambda + z^\ell \nabla_\theta(\Lambda) = \Lambda + z^\ell \tilde{\nabla}_\theta(\Lambda)$.*

Proof. Let (e) be a Smith basis of Λ with respect to M. Let $A = \mathrm{Mat}(\nabla_\theta, (e))$ and $\tilde{A} = \mathrm{Mat}(\tilde{\nabla}_\theta, (e))$. Let $\mathcal{K} = (k_1, \dots, k_n)$ be the elementary divisors of M in Λ. Then

$$A_{[z^{\mathcal{K}}]} - \tilde{A}_{[z^{\mathcal{K}}]} = z^{-\mathcal{K}}(A - \tilde{A})z^{\mathcal{K}} = \big((A - \tilde{A})_{ij} z^{k_j - k_i}\big).$$

Thus, we get

$$\begin{aligned} v(A_{[z^{\mathcal{K}}]} - \tilde{A}_{[z^{\mathcal{K}}]}) &= \min_{ij} v((A_{ij} - \tilde{A}_{ij}) z^{k_j - k_i}) \\ &= \min_{ij} \Big(v(A_{ij} - \tilde{A}_{ij}) + k_j - k_i\Big) \\ &\geqslant \min_{ij} v(A_{ij} - \tilde{A}_{ij}) + \min_{ij}(k_j - k_i) = v(A - \tilde{A}) - \delta(\Lambda, M). \end{aligned}$$

Therefore, the valuation of $A_{[z^{\mathcal{K}}]} - \tilde{A}_{[z^{\mathcal{K}}]}$ is at least $T' = T - \delta(\Lambda, M)$. The result ii) follows from Lemma 3.11 iii), since $\delta(\Lambda + z^\ell \nabla_\theta(\Lambda), \Lambda) = \mathfrak{p}_\Lambda(z^\ell \nabla) = \mathfrak{p}_\Lambda(\nabla) - \ell$.

Assume now that $\nabla \underset{-\ell-1}{\overset{\Lambda}{\sim}} \tilde{\nabla}$. Let $u \in \Lambda + z^\ell \nabla_\theta(\Lambda)$. There exist $v, w \in \Lambda$ such that $u = v + z^\ell \nabla_\theta(w)$. Since we can write $u = v + z^\ell(\nabla_\theta - \tilde{\nabla}_\theta)(w) + z^\ell \tilde{\nabla}_\theta(w) \in \Lambda + z^{\ell-\ell-1+1}\Lambda + z^\ell \tilde{\nabla}_\theta(\Lambda) = \Lambda + z^\ell \tilde{\nabla}_\theta(\Lambda)$, we have $\Lambda + z^\ell \nabla_\theta(\Lambda) \subset \Lambda + z^\ell \tilde{\nabla}_\theta(\Lambda)$. The argument is symmetrical, hence $\Lambda + z^\ell \nabla_\theta(\Lambda) = \Lambda + z^\ell \tilde{\nabla}_\theta(\Lambda)$. □

Proposition 4.6. *Let ∇ and $\tilde{\nabla}$ be connections on V, and Λ a lattice in V. Let $\mathfrak{p} = \mathfrak{p}_\Lambda(\nabla)$ and $\kappa = \kappa(\nabla)$. If $\nabla \underset{T}{\overset{\Lambda}{\sim}} \tilde{\nabla}$ holds with $T \geqslant \frac{n(n-1)}{2}(\mathfrak{p}-\kappa)$ then*

i) $\Lambda_K(\nabla) = \Lambda_K(\tilde{\nabla})$,

ii) $\nabla \underset{T'}{\overset{\Lambda_K(\nabla)}{\sim}} \tilde{\nabla}$ *where* $T' = T - \frac{n(n-1)}{2}(\mathfrak{p}-\kappa)$.

Proof. Since we assume $\rho(\nabla) = 1$, according to Proposition 3.12, the Katz lattice is given by

$$\Lambda_K(\nabla) = \left(\mathcal{F}^{n-1}_{z^\kappa\theta}(\Lambda^*)\right)^*$$

where κ is equal to the minimal Poincaré rank. It is obvious that $\nabla \underset{T}{\overset{\Lambda}{\sim}} \tilde{\nabla} \Leftrightarrow \nabla^* \underset{T}{\overset{\Lambda^*}{\sim}} \tilde{\nabla}^*$, hence, we will prove that $\mathcal{F}^{n-1}_{z^\kappa\nabla_\theta}(\Lambda) = \mathcal{F}^{n-1}_{z^\kappa\tilde{\nabla}_\theta}(\Lambda)$.

Put $\mathcal{F}^k = \mathcal{F}^k_{z^\kappa\nabla_\theta}(\Lambda)$ and $\tilde{\mathcal{F}}^k = \mathcal{F}^k_{z^\kappa\tilde{\nabla}_\theta}(\Lambda)$ for all $k \geqslant 0$. We have the following increasing chain of lattices

$$\Lambda = \mathcal{F}^0 \subset \mathcal{F}^1 \subset \cdots \subset \mathcal{F}^k \subset \mathcal{F}^{k+1} = \mathcal{F}^k + z^\kappa\nabla_\theta(\mathcal{F}^k) \subset \cdots \subset \mathcal{F}^{n-1}.$$

According to Lemma 3.11 iii), we have $v_{\mathcal{F}^{n-1}}(\Lambda) = 0$ hence $v_{\mathcal{F}^{k+\ell}}(\mathcal{F}^k) = 0$ for any $\ell \geqslant 0$. Since $T \geqslant -\kappa - 1$, it follows from Lemma 4.5 iii), that $\mathcal{F}^1 = \tilde{\mathcal{F}}^1$ and from Lemma 4.5 ii), that $\nabla \underset{T_1}{\overset{\mathcal{F}^1}{\sim}} \tilde{\nabla}$ where $T_1 = T - (\mathfrak{p}-\kappa)$. Assume now for $k < n-1$ that $\mathcal{F}^k = \tilde{\mathcal{F}}^k$ and $\nabla \underset{T_k}{\overset{\mathcal{F}^k}{\sim}} \tilde{\nabla}$. In a similar way, we have $\nabla \underset{T_{k+1}}{\overset{\mathcal{F}^{k+1}}{\sim}} \tilde{\nabla}$, hence

$$T_{k+1} = T_k - \delta(\mathcal{F}^k, \mathcal{F}^{k+1}) = T - \sum_{i=0}^{k} \delta(\mathcal{F}^i, \mathcal{F}^{i+1}).$$

According to Lemma 2.3, we have

$$\delta(\mathcal{F}^i, \mathcal{F}^{i+1}) = k_{n,\mathcal{F}^{i+1}}(\mathcal{F}^i) \leqslant k_{n,\mathcal{F}^{i+1}}(\Lambda) = -v_\Lambda(\mathcal{F}^{i+1}).$$

Since $\mathcal{F}^{i+1}$ is spanned by the images of a basis of Λ under iterations of the operator $z^\kappa\nabla_\theta$ we have $-v_\Lambda(\mathcal{F}^{i+1}) \leqslant (i+1)(\mathfrak{p}-\kappa)$. Accordingly, we get

$$T_{k+1} \geqslant T - \sum_{i=0}^{k}(i+1)(\mathfrak{p}-\kappa) = T - \frac{(k+1)(k+2)}{2}(\mathfrak{p}-\kappa) \geqslant 0 > -\kappa - 1.$$

Thus we have $\mathcal{F}^{k+1} = \tilde{\mathcal{F}}^{k+1}$. Therefore, the procedure can be carried until $n-1$. Consequently, $\mathcal{F}^{n-1} = \tilde{\mathcal{F}}^{n-1}$, and since $\delta(\Lambda, \mathcal{F}^{n-1}) \leqslant \frac{n(n-1)}{2}(\mathfrak{p}-\kappa)$, we find that $\nabla \underset{T'}{\overset{\mathcal{F}^{n-1}}{\sim}} \tilde{\nabla}$. By duality, we get the stated result. $\square$

4.2 Estimates for the coefficients

In this section, we investigate at what rank we may truncate the connection matrix without losing important information. For any integer T, let us denote with $A_{|T}$ the truncation of A at order T

$$A_{|T} = \frac{1}{z^p} A_{-p} + \frac{1}{z^{p-1}} A_{-p+1} + \cdots + A_T z^T.$$

Let $\mathrm{GL}_n(K)_0$ be the subset of matrices $P \in \mathrm{GL}_n(\mathcal{O})$ such that $v(P) = 0$.

Lemma 4.7. *Let A and $\tilde{A}$ be matrices in $\mathrm{M}_n(K)$. Let P and $\tilde{P}$ be gauge matrices in $\mathrm{GL}_n(K)_0$. Let $k = -v(P^{-1})$ and $p = -v(A)$. Then the following holds.*

i) *If $P - \tilde{P} \in z^{T+k}\mathrm{M}_n(\mathcal{O})$ then $P^{-1}\tilde{P} - I \in z^T\mathrm{M}_n(\mathcal{O})$. In particular, if $T \geqslant 1$, we have $P^{-1}\tilde{P} \in \mathrm{GL}_n(\mathcal{O})$ and $v(\tilde{P}^{-1}) = v(P^{-1})$.*

ii) *Let $T \geqslant -k+1$. If $P - \tilde{P} \in z^{T+2k}\mathrm{M}_n(\mathcal{O})$ then $P^{-1} - \tilde{P}^{-1} \in z^T\mathrm{M}_n(\mathcal{O})$.*

iii) *For any integer $T \geqslant -p+1$, if $P - \tilde{P} \in z^{T+2k+p}\mathrm{M}_n(\mathcal{O})$ then $A_{[P]} - A_{[\tilde{P}]} \in z^T\mathrm{M}_n(\mathcal{O})$.*

iv) *If $A - \tilde{A} \in z^{T+k}\mathrm{M}_n(\mathcal{O})$, then $A_{[P]} - \tilde{A}_{[P]} \in z^T\mathrm{M}_n(\mathcal{O})$.*

v) *$A_{[P]} - (A_{|T+k})_{[P_{|T+2k+p}]} \in z^{T+1}\mathrm{M}_n(\mathcal{O})$.*

vi) *For any positive integer T, one has $(P\tilde{P})_{|T} - P_{|T}\tilde{P}_{|T} \in z^{T+1}\mathrm{M}_n(\mathcal{O})$.*

Proof. i) We have $P^{-1}\tilde{P} = P^{-1}(P + (\tilde{P} - P)) = I - P^{-1}(P - \tilde{P})$. Hence

$$v(P^{-1}\tilde{P} - I) = v(P^{-1}(P - \tilde{P})) \geqslant T + k - k = T.$$

If $T \geqslant 1$, we get $P^{-1}\tilde{P} \in \mathrm{GL}_n(\mathcal{O})$.

ii) One has $P^{-1} - \tilde{P}^{-1} = P^{-1}(\tilde{P} - P)\tilde{P}^{-1}$. Accordingly, we get

$$v(P^{-1} - \tilde{P}^{-1}) \geqslant v(P^{-1}) + v(\tilde{P} - P) + v(\tilde{P}^{-1}).$$

However, the previous result ensures that $v(\tilde{P}^{-1}) = v(P^{-1}) = -k$ holds. Hence $v(P^{-1} - \tilde{P}^{-1}) \geqslant T + 2k - 2k = T$.

iii) We have $A_{[P]} - A_{[\tilde{P}]} = P^{-1}AP - P^{-1}\theta P - \tilde{P}^{-1}A\tilde{P} - \tilde{P}^{-1}\theta\tilde{P}$. According to what has just been proved, write $P^{-1}\tilde{P} = I + z^\alpha U$ with $U \in \mathrm{M}_n(\mathcal{O})$ where $\alpha = T + k + p$. Then $\tilde{P}^{-1}P = I + z^\alpha\tilde{U}$ with $\tilde{U} \in \mathrm{M}_n(\mathcal{O})$ also holds. By replacing we get

$$\begin{aligned}
A_{[\tilde{P}]} &= (I + z^\alpha\tilde{U})P^{-1}AP(I + z^\alpha U) - (I + z^\alpha\tilde{U})P^{-1}\theta(P(I + z^\alpha U)) \\
&= P^{-1}AP + z^\alpha\tilde{U}P^{-1}AP + z^\alpha P^{-1}APU + z^{2\alpha}\tilde{U}P^{-1}APU \\
&\quad - (I + z^\alpha\tilde{U})P^{-1}(\theta P + z^\alpha\theta PU + \alpha z^\alpha PU + z^\alpha P\theta U))
\end{aligned}$$

$$= A_{[P]} + z^{\alpha}\big(\tilde{U}P^{-1}AP + P^{-1}APU - \tilde{U}P^{-1}\theta PP^{-1}\theta PU - \alpha U - \theta U\big) \\ + z^{2\alpha}\big(\tilde{U}P^{-1}APU - \tilde{U}P^{-1}\theta PU - \alpha\tilde{U}U - \tilde{U}\theta U\big).$$

Hence, we find

$$A_{[P]} - A_{[\tilde{P}]} = z^{\alpha}\big(\tilde{U}P^{-1}\theta P + P^{-1}\theta PU + \alpha U + \theta U - \tilde{U}P^{-1}AP - P^{-1}APU\big) \\ + z^{2\alpha}\big(\tilde{U}P^{-1}\theta PU + \alpha\tilde{U}U + \tilde{U}\theta U - \tilde{U}P^{-1}APU\big).$$

Accordingly, we see that

$$v\big(A_{[P]} - A_{[\tilde{P}]}\big) \geqslant \min(\alpha - k - p, 2\alpha - k - p) = \alpha - k - p = T + k + p - k - p = T.$$

iv) One has $A_{[P]} - \tilde{A}_{[P]} = P^{-1}(A - \tilde{A})P$. We have then

$$v\big(A_{[P]} - \tilde{A}_{[P]}\big) \geqslant v(P^{-1}) + v(A - \tilde{A}) + v(P) = -k + T + k = T.$$

v) We have

$$A_{[P]} - (A_{|T+k})_{[P_{|T+2k+p}]} = A_{[P]} - (A_{|T+k})_{[P]} + (A_{|T+k})_{[P]} - (A_{|T+k})_{[P_{|T+2k+p}]}.$$

According to iii) and iv)), we have the announced result.

vi) The last statement is a simple calculation, and is left to the reader. □

Corollary 4.8. *Let Λ be a lattice in V. For any basis (e) of Λ, any basis (ε) of its Levelt lattice Λ_L, the attached matrix $A = \mathrm{Mat}(\nabla_\theta, (e)) \in z^{-p}\mathrm{M}_n(\mathcal{O})$ of ∇_θ over Λ and the gauge $P = P_{(e),(\varepsilon)}$ satisfy*

$$A_{[P]} - (A_{|T_0})_{[P_{|2T_0+p}]} \in z\mathrm{M}_n(\mathcal{O})$$

where $T_0 = (n-1)p$. In particular, we can compute the exponents of the system of matrix A with a finite number of calculations using only the first $np + 1$ coefficients of A.

Proof. According to v) of the previous lemma, $T_0 = -v(P^{-1}) = k_{n,\Lambda}(\Lambda_L)$ is sufficient. However, according to Corollary 3.27, we have $k_{n,\Lambda}(\Lambda_L) \leqslant (n-1)p$. Therefore the announced result holds with $T_0 = (n-1)p$. □

4.3 Standard notation

We now apply the previous methods to a system

$$\frac{dX}{dz} = A(z)X, \tag{4.1}$$

where $A(z) \in \mathrm{M}_n(K)$ is the matrix of the differential operator ∇_θ in a basis (e) of $V = K^n$. We only consider gauge transforms with respect to the derivative θ

$$A_{[P]} = P^{-1}AP - P^{-1}\theta P.$$

Since all the lattices introduced in the previous section are either maximal or minimal with respect to the original one, all gauge transforms P occurring in this section either satisfy $v(P) = 0$ or $v(P^{-1}) = 0$. Much in the same way, we assume that the Poincaré rank of the system shall never increase by gauge transformations. Let us fix the following *standard notation*.

- (e) a basis,
- $A = \mathrm{Mat}(\nabla_\theta, (e)) = \frac{1}{z^p} A_{-p} + \frac{1}{z^{p-1}} A_{-p+1} + \cdots$ the connection matrix,
- $P = P_0 + P_1 z + \cdots \in \mathrm{M}_n(\mathcal{O}) \cap \mathrm{GL}_n(K)$ the gauge matrix we apply,
- (ε) the basis obtained under the gauge P,
- $B = A_{[P]} = \mathrm{Mat}(\nabla_\theta, (\varepsilon)) = \frac{1}{z^p} B_{-p} + \frac{1}{z^{p-1}} B_{-p+1} + \cdots$ the transformed connection matrix.
- $\Lambda = \mathcal{L}(e)$ and $M = \mathcal{L}(\varepsilon)$ the spanned lattices.

Let A_{irr} denote the irregular part

$$A_{\mathrm{irr}} = \frac{1}{z^p} A_{-p} + \frac{1}{z^{p-1}} A_{-p+1} + \cdots + \frac{1}{z} A_{-1}$$

of the matrix A.

The gauge equation $\theta P = AP - PB$ gives in expanded form the following system of matrix equations :

$$\left.\begin{array}{lrcl}
(0) & A_{-p}P_0 - P_0 B_{-p} & = & 0 \\
(1) & A_{-p}P_1 + A_{-p+1}P_0 - P_1 B_{-p} - P_0 B_{-p+1} & = & 0 \\
\vdots & & \vdots & \\
(p) & A_{-p}P_p + \cdots + A_0 P_0 - P_p B_{-p} - \cdots - P_0 B_0 & = & 0 \\
(p+1) & A_{-p}P_{p+1} + \cdots + A_1 P_0 - P_{p+1} B_{-p} - \cdots - P_0 B_1 & = & P_1 \\
\vdots & & \vdots & \\
(p+k) & A_{-p}P_{p+k} + \cdots + A_k P_0 - P_{p+k} B_{-p} - \cdots - P_0 B_k & = & kP_k
\end{array}\right\} (S)$$

Note that if $P \notin \mathrm{GL}_n(\mathcal{O})$, the first equation for instance does not wholly define B_{-p}, and that further equations are needed.

4.4 Computation of exponents on a compatible lattice

The notion of compatible lattice plays for an irregular connection, the same rôle as stable lattices for regular connections. Unlike the regular case, it is not obvious, for an irregular connection, even on a compatible lattice, to determine the exponents directly from A. It is necessary to separate the determinant map from the regular attached connection.

Let us consider the standard notation. Let $q \leqslant p$ be an integer, and let us denote with (C_q) the following system of matrix commutation relations.

$$(C_q)\begin{cases}[P_0, A_{-p}] = \cdots = [P_0, A_{-p+q}] = 0\\ \vdots\\ [P_q, A_{-p}] = 0,\end{cases}$$

Lemma 4.9. *Under our standard notation, assume that the commutation relations (C_q) are satisfied with a gauge $P \in \mathrm{GL}_n(\mathcal{O})$. Then the first $q+2$ equations of (S) can be simplified as follows.*

$$\left.\begin{array}{ll}(0) & B_{-p} = A_{-p}\\ \vdots & \vdots\\ (q) & B_{-p+q} = A_{-p+q}\\ (q+1) & B_{-p+q+1} = P_0^{-1}A_{-p+q+1}P_0 + \displaystyle\sum_{k=1}^{q+1}[A_{-p+q+1-k}, P_0^{-1}P_k]\end{array}\right\}(S_q)$$

Proof. We proceed by induction on q. For $q = 0$, equation (0) can be written as $B_{-p} = P_0^{-1}A_{-p}P_0$. Since $[P_0, A_{-p}] = 0$, one gets $B_{-p} = A_{-p}$. Equation (1) is then

$$\begin{aligned}B_{-p+1} &= P_0^{-1}A_{-p+1}P_0 + P_0^{-1}(A_{-p}P_1 - P_1B_{-p})\\ &= P_0^{-1}A_{-p+1}P_0 + P_0^{-1}[A_{-p}, P_1]\\ &= P_0^{-1}A_{-p+1}P_0 + [A_{-p}, P_0^{-1}P_1]\end{aligned}$$

once again thanks to the condition $[P_0, A_{-p}] = 0$.

Consider $k < q$. Assume that the conditions (C_k) are satisfied, and that the equations of system (S_{k-1}) also hold. We want to establish that system (S_k) holds. By assumption, equation (k) of (S) can be written as follows.

$$B_{-p+k} = P_0^{-1}A_{-p+k}P_0 + \sum_{\ell=1}^{k}[A_{-p+k-\ell}, P_0^{-1}P_\ell].$$

But, since the conditions (C_k) are assumed to be satisfied, the Lie brackets are all equal to zero, whereas P_0 is assumed to commute with A_{-p+k}. Hence, one simply gets

$$B_{-p+k} = A_{-p+k}.$$

The matrix P_0 is invertible, hence equation $(k+1)$ of system (S) can be written as

$$B_{-p+k+1} = P_0^{-1}\Big(A_{-p+k+1}P_0 + \sum_{\ell=1}^{k+1}(A_{-p+k+1-\ell}P_\ell - P_\ell B_{-p+k+1-\ell})\Big).$$

But since we proved that $B_\ell = A_\ell$ for $\ell = -p, \dots, -p+k$, we get

$$B_{-p+k+1} = P_0^{-1} A_{-p+k+1} P_0 + P_0^{-1} \Big(\sum_{\ell=1}^{k+1} [A_{-p+k+1-\ell}, P_\ell] \Big).$$

Now, since $[A_{-p+\ell}, P_0] = 0$, for $\ell \leqslant k$, we get

$$P_0^{-1}[A_{-p+k+1-\ell}, P_\ell] = [A_{-p+k+1-\ell}, P_0^{-1} P_\ell],$$

hence

$$B_{-p+k+1} = P_0^{-1} A_{-p+k+1} P_0 + \Big(\sum_{\ell=1}^{k+1} [A_{-p+k+1-\ell}, P_0^{-1} P_\ell] \Big)$$

which ends the proof. □

Let us introduce a little matrix machinery. Consider an $n \times n$ matrix A. Let $\mathcal{B} = (A_{IJ})$ and $\tilde{\mathcal{B}} = (A_{KL})$ be two block-partitions of A. The partition $\tilde{\mathcal{B}}$ is said to be *finer* than $\mathcal{B}$ if any block of $\tilde{\mathcal{B}}$ is contained in a block of $\mathcal{B}$. This is obviously a partial ordering on the block-partitions of A. The *intersection* of the two block-partitions $\mathcal{B}$ and $\tilde{\mathcal{B}}$ is the largest block-partition $\mathcal{B} \wedge \tilde{\mathcal{B}}$ which is finer than both $\mathcal{B}$ and $\tilde{\mathcal{B}}$. For a diagonal matrix, call *suitable partition* the largest block-partition according to the grouping of consecutive equal eigenvalues.

Lemma 4.10. *Let* $A(z) = \mathrm{Mat}(\nabla_\theta, (e)) = \frac{1}{z^p} A_{-p} + \frac{1}{z^{p-1}} A_{-p+1} + \cdots$ *be the matrix of* ∇_θ *in the basis* (e)*. Let* $0 \leqslant q \leqslant p$ *be an integer. Assume that* $A_{-p}, \dots, A_{-p+q-1}$ *are diagonal matrices. If the lattice* $\Lambda = \mathcal{L}(e)$ *is compatible with* ∇*, then there exists a gauge transform* $P = P_0 + \cdots + P_q z^q \in \mathrm{GL}_n(\mathcal{O}))$ *such that one has*

$$\left.\begin{aligned} [A_{-p}, P_0] = \cdots = [A_{-p+q-1}, P_0] &= 0 \\ &\vdots \\ [A_{-p}, P_{q-1}] &= 0, \end{aligned}\right\} (C_{q-1})$$

and the matrix $B = A_{[P]}$ *satisfies*

$$B = \frac{1}{z^p} A_{-p} + \cdots + \frac{1}{z^{p-q+1}} A_{-p+q-1} + \frac{1}{z^{p-q}} B_{-p+q} + \cdots$$

where B_{-p+q} *is a diagonal matrix if* $q < p$*, or* B_{-p+q} *is a matrix commuting with* $A_{-p}, \dots, A_{-1}$ *if* $q = p$*.*

Proof. Consider the system (S) of gauge equations. Let $\mathcal{B}_{-p}, \dots, \mathcal{B}_{-p+q-1}$ be the suitable partitions of the matrices $A_{-p}, \dots, A_{-p+q-1}$ and $\mathcal{B} = \mathcal{B}_{-p} \wedge \cdots \wedge \mathcal{B}_{-p+q-1}$ their intersection. Let $(M_{IJ}^{(k)})$ denote the block-partitioning of the matrix M_k according to $\mathcal{B}$. For a diagonal matrix A_k let $\lambda_I^{(k)}$ be the only eigenvalue appearing in the diagonal block of index I.

We proceed by induction on q. If $q = 0$, the assumption on A is empty. Since we assume the lattice Λ to be compatible, the matrix A_{-p} can be diagonalized. Indeed, the matrix A_{-p} is the first coefficient in the expansion of the determinant map φ. Therefore, there exists $P_0 \in \mathrm{GL}_n(\mathcal{O})$ such that

$$B_{-p} = P_0^{-1} A_{-p} P_0 \tag{4.2}$$

is diagonal. However, equation (4.2) is nothing but equation (0) of the system (S), hence the lemma is proved for $q = 0$.

Assume now that the previous procedure has achieved the assumptions of the proposition for all $q' \leqslant q$. Equations from (0) to $(q-1)$ of system (S) are satisfied with diagonal matrices $B_{-p} = A_{-p}, \dots B_{-p+q-1} = A_{-p+q-1}$ and a gauge $P = P_0 + \cdots + P_{q-1}z^{q-1}$ that satisfies the commutation equations. We are going to modify P into a gauge $\tilde{P} = \tilde{P}_0 + \cdots + \tilde{P}_q z^q$ such that equations from (0) to $(q-1)$ are still satisfied with the same matrices, and such that equation (q) can be solved with a diagonal matrix B_{-p+q}.

Due to the commutation assumption, the matrix P_k is $\mathcal{B}_{-p} \wedge \cdots \wedge \mathcal{B}_{-p+q-1-k}$-block diagonal, for $k = 0, \dots, q-1$, i.e. P satisfies system (C_{q-1}). Then, thanks to Lemma 4.9, equation (q) can be written as

$$\begin{aligned} B_{-p+q} = P_0^{-1} A_{-p+q} P_0 &+ [A_{-p+q-1}, P_0^{-1} P_1] \\ &+ [A_{-p+q-2}, P_0^{-1} P_2] + \cdots + [A_{-p}, P_0^{-1} P_q]. \end{aligned}$$

Written according to the $\mathcal{B}$-block partition, this becomes

$$B_{IJ}^{(-p+q)} = P_{II}^{(0)^{-1}} A_{IJ}^{(-p+q)} P_{JJ}^{(0)} + P_{II}^{(0)^{-1}} \sum_{k=1}^{q} P_{IJ}^{(k)} \big(\lambda_I^{(-p+q-k)} - \lambda_J^{(-p+q-k)}\big). \tag{4.3}$$

For the indices (I, I) that correspond to the $\mathcal{B}$-diagonal blocks, all the differences $\lambda_I^{(-p)} - \lambda_I^{(-p)}, \dots, \lambda_I^{(-p+q)} - \lambda_I^{(-p+q)}$ are equal to zero, hence the equation breaks down to

$$B_{II}^{(-p+q)} = P_{II}^{(0)^{-1}} A_{II}^{(-p+q)} P_{II}^{(0)}.$$

If $q < p$, there is a solution to this equation with a diagonal matrix $B_{II}^{(-p+q)}$, therefore the matrix $A_{II}^{(-p+q)}$ is diagonalizable. Choose then a matrix $\tilde{P}_{II}^{(0)}$ which diagonalizes the block $A_{II}^{(-p+q)}$. If $q = p$, one can choose $\tilde{P}_{II}^{(0)}$ at random, for example, one can ask that it puts $A_{II}^{(-p+q)}$ in Jordan normal form.

For a bi-index (I, J) such that $I \neq J$, denote with ℓ the smallest integer such that $\lambda_I^{(-p+\ell)} - \lambda_J^{(-p+\ell)} \neq 0$. Then (I, J) lies inside the $\mathcal{B}_{-p} \wedge \cdots \wedge \mathcal{B}_{-p+\ell-1}$-diagonal, but outside the $\mathcal{B}_{-p+\ell}$-block diagonal. Then, one has

$$P_{IJ}^{(0)} = \cdots = P_{IJ}^{(q-\ell-1)} = 0.$$

On the other hand, one also has $\lambda_I^{(-p)} = \lambda_J^{(-p)}, \dots, \lambda_I^{(-p+\ell-1)} = \lambda_J^{(-p+\ell-1)}$. The equation (4.3) becomes then simply

$$B_{IJ}^{(-p+q)} = P_{II}^{(0)^{-1}} A_{IJ}^{(-p+q)} P_{JJ}^{(0)} + P_{II}^{(0)^{-1}} P_{IJ}^{(q-\ell)} \left(\lambda_I^{(-p+\ell)} - \lambda_J^{(-p+\ell)}\right). \tag{4.4}$$

We then solve the block equation with

$$B_{IJ}^{(-p+q)} = 0$$

by letting

$$\tilde{P}_{IJ}^{(q-\ell)} = \frac{1}{\lambda_J^{(-p+\ell)} - \lambda_I^{(-p+\ell)}} A_{IJ}^{(-p+q)} P_{JJ}^{(0)}. \tag{4.5}$$

The remaining blocks must satisfy

$$\tilde{P}_{IJ}^{(0)} = \cdots = \tilde{P}_{IJ}^{(q-\ell-1)} = 0.$$

We can moreover ask that

$$\tilde{P}_{IJ}^{(q-\ell+1)} = \cdots = \tilde{P}_{IJ}^{(q)} = 0.$$

Putting all the $\mathcal{B}$-blocks $\tilde{P}_{IJ}^{(k)}$ together, one gets a gauge $\tilde{P}$, which satisfies the commutation conditions (C_{q-1}), and such that if $q < p$, the matrix $B = A_{[\tilde{P}]}$ has a diagonal coefficient B_{-p+q}, and, if $q = p$, the matrix B_0 has non-zero elements only inside the $\mathcal{B} = \mathcal{B}_{-p} \wedge \cdots \wedge \mathcal{B}_{-1}$-block diagonal, hence it commutes with all the matrices $A_{-p}, \ldots, A_{-1}$. This finishes our induction. □

Lemma 4.11. *Conversely, assume that $A_{\mathrm{irr}} = B_{\mathrm{irr}}$ and that they are both diagonal matrices. Then the gauge $P \in \mathrm{GL}_n(\mathcal{O})$ such that $B = A_{[P]}$ satisfies the conditions (C_{p-1}).*

Proof. We are going to prove by induction that if $A_{|-p+q} = B_{|-p+q}$ then the gauge P satisfies conditions (C_q). For $q = 0$, one has equation (0)

$$A_{-p} = P_0^{-1} A_{-p} P_0.$$

Therefore we get $[A_{-p}, P_0] = 0$ which is condition (C_0).

Assume now that $A_{|-p+q} = B_{|-p+q}$ and that P satisfies (C_{q-1}). We want to prove that P satisfies then (C_q). Writing the last equation (q) according to the $\mathcal{B}$-block decomposition, one gets

$$A_{II}^{(-p+q)} = P_{II}^{(0)^{-1}} A_{II}^{(-p+q)} P_{II}^{(0)}$$

inside the $\mathcal{B}$-block diagonal, which naturally holds since $A_{II}^{(-p+q)}$ are scalar matrices. For the indices outside the block diagonal, we get

$$A_{IJ}^{(-p+q)} = P_{II}^{(0)^{-1}} A_{IJ}^{(-p+q)} P_{JJ}^{(0)} + P_{II}^{(0)^{-1}} P_{IJ}^{(q-\ell)} \left(\lambda_I^{(-p+\ell)} - \lambda_J^{(-p+\ell)}\right)$$

where ℓ is the smallest index such that $\lambda_I^{(-p+\ell)} - \lambda_J^{(-p+\ell)} \neq 0$. According to the assumption (C_{q-1}), we have $P_{IJ}^{(q-k)} = 0$ for all $\ell+1 \leqslant k \leqslant q$. Since both $B_{IJ}^{(-p+q)} =$

0 and $A_{IJ}^{(-p+q)} = 0$, one gets $P_{IJ}^{(q-\ell)} = 0$. Thus the matrix $P_{q-\ell}$ commutes with $A_{-p}, \ldots, A_{-p+\ell}$, which means that P satisfies (C_q). Therefore, the lemma is proved, since $A_{\mathrm{irr}} = B_{\mathrm{irr}}$ corresponds to $q = p - 1$. □

We now prove that the irregular part of the connection matrix on Λ is essentially unique. We first need a small lemma of elementary matrix algebra.

Lemma 4.12. *Let D and Δ be conjugate diagonal matrices, and $P \in \mathrm{GL}_n(\mathbb{C})$ such that $\Delta = P^{-1}DP$. Then there exist $T, \tilde{P} \in \mathrm{GL}_n(\mathbb{C})$ such that $P = \tilde{P}T$ and*

i) *T is a permutation matrix,*

ii) $[D, \tilde{P}] = 0$.

Proof. D and Δ have the same eigenvalues, hence there exists a permutation $T \in \mathrm{GL}_n(\mathbb{C})$ such that $\Delta = T^{-1}DT$. Since $D = TP^{-1}DPT^{-1}$, we have $[D, PT^{-1}] = 0$ hence $\tilde{P} = PT^{-1}$ matches the requirements of the lemma. □

Lemma 4.13. *Let $\Lambda = \mathcal{L}(e)$ be a compatible lattice. Consider the standard notation. Assume that A_{irr} and B_{irr} are both diagonal matrices. Then there exists a permutation matrix $T \in \mathrm{GL}_n(\mathbb{C})$ such that $(A_{[T]})_{\mathrm{irr}} = B_{\mathrm{irr}}$.*

Proof. We also use induction on the number q such that $A_{|-p+q} = B_{|-p+q}$. For $q = 0$, equation (0) is as always

$$B_{-p} = P_0^{-1} A_{-p} P_0.$$

Since both matrices are diagonal, there exist $T_0, \tilde{P}_0 \in \mathrm{GL}_n(\mathbb{C})$ such that T_0 is a permutation matrix, $[A_{-p}, \tilde{P}] = 0$ and $P_0 = \tilde{P}_0 T_0$. Accordingly, one has

$$B_{-p} = T_0^{-1} \tilde{P}_0^{-1} A_{-p} \tilde{P}_0 T_0 = T_0^{-1} A_{-p} T_0.$$

This means that $(A_{[T_0]})_{|-p} = B_{|-p}$.

Now assume that the first $q-1$ coefficients of A_{irr} and B_{irr} are equal. Then the gauge P satisfies conditions (C_{q-1}), since the coefficients $A_{-p}, \ldots, A_{-p+q-1}$ are diagonal. Writing the equation (q) of system (S) according to the $\mathcal{B}_{-p} \wedge \cdots \wedge \mathcal{B}_{-p+q}$-block decomposition, one gets

$$B_{II}^{(-p+q)} = P_{II}^{(0)^{-1}} A_{II}^{(-p+q)} P_{II}^{(0)}$$

inside the diagonal, from whence we deduce that there exists a permutation matrix $T_{II}^{(q)}$ such that $T^{(q)}{}_{II}^{-1} A_{II}^{(-p+q)} T_{II}^{(q)} = B_{II}^{(-p+q)}$. For the indices out of the block diagonal, we get

$$B_{IJ}^{(-p+q)} = P_{II}^{(0)^{-1}} A_{IJ}^{(-p+q)} P_{JJ}^{(0)} + P_{II}^{(0)^{-1}} P_{IJ}^{(q-\ell)} \left(\lambda_I^{(-p+\ell)} - \lambda_J^{(-p+\ell)}\right)$$

where ℓ is the smallest index such that $\lambda_I^{(-p+\ell)} - \lambda_J^{(-p+\ell)} \neq 0$. Since both $B_{IJ}^{(-p+q)} = 0$ and $A_{IJ}^{(-p+q)} = 0$, one has $P_{q-\ell} = 0$. Hence the matrix $P_{q-\ell}$ commutes with

$A_{-p}, \ldots, A_{-p+\ell}$, i.e. P satisfies (C_q). Call T_q the matrix obtained by the recollecting of all blocks $T_{II}^{(q)}$. Clearly, the matrix T_q is a permutation matrix that commutes with $A_{-p}, \ldots, A_{-p+q-1}$ and the matrix $\tilde{A} = A_{[T_q]}$ satisfies $\tilde{A}_{|-p+q} = B_{|-p+q}$. Hence the induction is finished. Note that the matrix T of the lemma is just the product $T_0 T_1 \ldots T_{p-1}$ and is therefore a permutation matrix. $\square$

Proposition 4.14. *Let* $A(z) = \mathrm{Mat}(\nabla_\theta, (e)) = \frac{1}{z^p} A_{-p} + \frac{1}{z^{p-1}} A_{-p+1} + \cdots$ *be the matrix of* ∇_θ *in the basis* (e)*. Assume the lattice* $\Lambda = \mathcal{L}(e)$ *is compatible with* ∇*. Then*

1. *there exists a gauge transform* $P = P_0 + \cdots + P_p z^p \in \mathrm{GL}_n(\mathbb{C}[z])$ *such that the matrix* $B = A_{[P]}$ *satisfies*
   a) B_{irr} *is diagonal,*
   b) $[B_0, B_{-k}] = 0$ *for* $k = 1, \ldots, p$*;*
2. *for any matrix* $\tilde{B}$ *which is* $\mathrm{GL}_n(\mathcal{O})$*-equivalent to* A *and such that conditions* 1.a) *and* 1.b) *hold, the matrices* B_0 *and* $\tilde{B}_0$ *are conjugate.*

Moreover, the exponents of the system are the eigenvalues of the matrix B_0.

Proof. Assertion 1. of the proposition is a direct consequence of Lemma 4.10. One can perform a gauge P of degree at most p such that B_{irr} is diagonal and such that $[B_{-k}, B_0] = 0$ for all $k = 1, \ldots, p$.

The lattice Λ is compatible, hence, according to Proposition 3.5, there exists a BVG normal form

$$\mathfrak{A} = D_{-p} z^{-p} + \cdots + D_{-1} z^{-1} + N + z^N L z^{-N}.$$

Denote with $N + L_0$ the matrix $\mathfrak{A}_0$. We will prove statement 2. if we prove that for any matrix B satisfying conditions 1.a) and 1.b), the matrix B_0 is conjugate to $N + L_0$. Let such a B be given. According to Lemma 4.13, there exists a permutation matrix $T \in \mathrm{GL}_n(\mathbb{C})$ such that

$$\tilde{B} = T^{-1} B T = D_{-p} z^{-p} + \cdots + D_{-1} z^{-1} + \tilde{B}_0 + \cdots$$

Of course, $\tilde{B}$ also satisfies conditions 1.a) and 1.b) since

$$[D_{-k}, \tilde{B}_0] = [T^{-1} B_{-k} T, T^{-1} B_0 T] = 0 \quad \text{for all } k = 1, \ldots, p.$$

There is a gauge P such that $\tilde{B}_{[P]} = \mathfrak{A}$. Consider the system (S) attached to the gauge P applied to $\tilde{B}$. Lemma 4.11 shows that the gauge P satisfies conditions (C_{p-1}). Equation (p) is then given in $\mathcal{B}$-blocks as

$$\mathfrak{A}_{II}^{(0)} = P_{II}^{(0)^{-1}} \tilde{B}_{II}^{(0)} P_{II}^{(0)}$$

and

$$\mathfrak{A}_{IJ}^{(0)} = P_{II}^{(0)^{-1}} \tilde{B}_{IJ}^{(0)} P_{JJ}^{(0)} + P_{II}^{(0)^{-1}} P_{IJ}^{(q-\ell)} \left(\lambda_I^{(-p+\ell)} - \lambda_J^{(-p+\ell)}\right)$$

where ℓ is the smallest index such that $\lambda_I^{(-p+\ell)} - \lambda_J^{(-p+\ell)} \neq 0$. Now, since both $\tilde{B}_0$ and $\mathfrak{A}_0$ commute with D_{-k} for all $k = 1, \ldots, p$, we have $\mathfrak{A}_{IJ}^{(0)} = \tilde{B}_{IJ}^{(0)} = 0$ for $I \neq J$ and thus, combining the $\mathcal{B}$-blocks, we get $\mathfrak{A}_0 = P_0^{-1} \tilde{B} P_0$. □

5 Algorithmic procedures

5.1 One-step saturation

Let the lattice Λ be given with a basis (e). Assuming that the derivation map τ satisfies $\tau(\mathcal{O}) \subset \mathcal{O}$, then the first saturated lattice $\Lambda + \nabla_\tau \Lambda$ is equal to the $\mathcal{O}$-module spanned by $(e_1, \ldots, e_n, \nabla_\tau e_1, \ldots, \nabla_\tau e_n)$. The core of the saturation algorithm is to extract an $\mathcal{O}$-basis of $\Lambda + \nabla_\tau \Lambda$ from those $2n$ vectors.

Lemma 5.1. *Denote with $A \in z^{-p}\mathrm{M}_n(\mathcal{O})$ the matrix $\mathrm{Mat}(\nabla_\tau, (e))$. Then the lattice $\Lambda + \nabla_\tau \Lambda$ is spanned in (e) by the columns of the $n \times 2n$ matrix*

$$(I_n \;\; A_{|-1}), \tag{5.1}$$

Proof. For any $u_1, \ldots, u_n \in \Lambda$, denote with $U \in \mathrm{M}_n(\mathcal{O})$ the matrix of its components in (e). The family $(e_1, \ldots, e_n, \nabla_\tau e_1 - u_1, \ldots, \nabla_\tau e_n - u_n)$ spans the same $\mathcal{O}$-module since the matrix of its components in $(e_1, \ldots, e_n, \nabla_\tau e_1, \ldots, \nabla_\tau e_n)$ is then $\begin{pmatrix} I_n & U \\ 0_n & I_n \end{pmatrix} \in \mathrm{GL}_{2n}(\mathcal{O})$. Hence, one can restrict the matrix A to its irregular part. □

Since A can be truncated to its irregular part, its coefficients can be assumed to be Laurent polynomials. Therefore we can perform the so-called Hermite normal form.

Proposition 5.2 (Hermite normal form). *For any $M \in \mathrm{M}_n(\mathbb{C}[z, z^{-1}])$, there exists an invertible matrix $U \in \mathrm{GL}_{2n}(\mathbb{C}[z])$ such that*

$$(I_n \;\; M)U = (0_n \;\; T), \tag{5.2}$$

where the columns of $T \in \mathrm{GL}_n(\mathbb{C}[z, z^{-1}])$ span the same $\mathbb{C}[z]$-module as those of $(I_n \;\; M)$.

We will say that T is a *Hermite normal form of M*.

Lemma 5.3. *If $A \in z^{-p}\mathrm{M}_n(\mathbb{C}[z, z^{-1}])$ has a block-diagonal structure, then there exists a Hermite normal form of A that has the same block-diagonal structure.*

Proof. Let

$$A = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_t \end{pmatrix}$$

be the block-diagonal structure of A, and M_{ij} the induced block structure of a matrix $M \in \mathrm{M}_n(\mathbb{C}[z, z^{-1}])$. In equation (5.2), write the matrix U as

$$U = \begin{pmatrix} P & R \\ Q & S \end{pmatrix}$$

where P, Q, R, S are square matrices. Equation $(I_n \;\; A)U = (0_n \;\; T)$ is equivalent to the following system:

$$(E_{ij}) \begin{cases} P_{ij} + A_i Q_{ij} = 0 \\ R_{ij} + A_i S_{ij} = T_{ij} \end{cases} \quad \text{for } 1 \leqslant i, j \leqslant t.$$

Proposition 5.2 shows that for any i, the system (E_{ii}) can be solved, whereas the redundant systems (E_{ij}) with $j \neq i$ can be assumed to be trivial. Therefore, we can find a Hermite normal form T which is block-diagonal, only by assuming that P, Q, R, and S are block-diagonal. Indeed, this simplification does not affect the invertibility of the total matrix U. □

Lemma 5.4. *Consider the derivation $\tau = z^\ell \theta$, with $\ell \geqslant m(\nabla)$. Let T be a Hermite normal form of $A_{|-1}$, and U be the corresponding gauge in $\mathrm{GL}_{2n}(\mathcal{O})$. Denote with $\tilde{T}$ the truncation $T_{|0}$ of T. Then there exists a matrix $P \in \mathrm{GL}_n(\mathcal{O})$ such that $\tilde{T} = TP$.*

Proof. Since $\ell \geqslant m(\nabla)$, Lemma 3.11 ensures us that $v(T^{-1}) = 0$. According to Lemma 4.4, one gets a $\mathrm{GL}_n(\mathcal{O})$-equivalent gauge by truncating T at its constant coefficient T_0. Hence there exists $P \in \mathrm{GL}_n(\mathcal{O})$ such that $\tilde{T} = TP$. □

Procedure *HNF* : Input A, output T.

(1) Let $t \in \mathbb{N}$ be the largest integer such that A has a $t \times t$ block-diagonal structure $A = \mathrm{diag}(A_1, \dots, A_t)$.

(2) **for** $i = 1$ to t **do**

$B_i := (A_i)_{|-1}$.

$S_i :=$ Hermite normal form (B_i).

$T_i := (S_i)_{|0}$.

(3) **end do**

(4) $T := \mathrm{diag}(T_1, \dots, T_t)$.

Remark 5.5. Note that this Lemma 5.4 gives us a sufficient condition for the minimal Poincaré rank procedure. Namely, denote with T the matrix obtained after procedure *HNF* applied to the matrix $\mathrm{Mat}(z^\ell \nabla_\theta, (e))$. If $v(T^{-1}) \neq 0$, then one can be sure that $\ell < m(\nabla)$.

5.2 The Katz lattice

In theory, one computes the Katz lattice after computing the minimal Poincaré rank. However, one can perform simultaneously both by simply computing the saturated lattice $\mathcal{F}^{n-1}_{z^k\nabla_\theta}(\Lambda)$ and by decreasing the integer k till the obtained lattice is no longer $z^k\nabla_\theta$-stable. However, since p can be much larger than the minimal m, we consider an additional feature that avoids a number of unnecessary computations. Recall that the Katz lattice is given by the saturated dual lattice $\Lambda_K(\nabla) = \mathcal{F}^{n-1}_{z^m\nabla^*_\theta}(\Lambda^*)$. Therefore, one must perform all the corresponding procedures with $\mathrm{Mat}(\nabla^*_\theta, (e^*)) = -{}^tA$.

Full saturation at rank ℓ. Given an order $0 \leqslant \ell \leqslant p-1$, one performs the following.

Procedure *SAT* : Input (A, ℓ), output $(stable, As, Ps)$;

(1) $B := z^\ell A$, and $i := 0$ (initializing).

(2) $i := i + 1$.

(3) $T := HNF(B)$.

(4) $S := T^{-1}$.

(5) **if** $v(S) > 0$ then *stable* := *NO*.

go to (8).

(6) **end if**

(7) $B := SBT - Sz^\ell\theta T$.

(8) **if** $v(B) \geqslant 0$ then output: *stable* := *YES*, $As := B$ and $Ps := T$.

if $i > n-1$ then output: *stable* := *NO* and $As := A$

(9) **else** go to (2).

(10) **end if**

This procedure computes recursively all lattices $\mathcal{F}^k_{z^\ell\theta}(\Lambda)$ from $k = 1$ to $n-1$, and breaks if we get a $z^\ell\nabla_\theta$-stable lattice before $n-1$ iterations, which is always possible. It also uses remark 5.5 to interrupt the algorithm if Λ has got non-zero valuation in its saturated lattice. It returns either the matrix of the system on a basis of the saturated lattice with the value *YES* meaning "$\ell \geqslant m(\nabla)$", or an "error message" with the value *NO* meaning "$\ell < m(\nabla)$" meaning that the index ℓ is too small. In this case, one must perform the procedure with another rank.

Minimal Poincaré rank. Since every $z^{\ell}\nabla_{\theta}$-stable lattice is also $z^{\ell+1}\nabla_{\theta}$-stable, the answer to the former procedure $SAT(A, \ell)$ gives a way to avoid checking either all ranks $k \geqslant \ell$, if the answer is a stable lattice, or all ranks $k \leqslant \ell$, if the answer is a non-stable lattice.

Considering an integer interval $I = \{\ell_1, \dots, \ell_2\}$, call ℓ_1 the lower end point $\mathrm{lep}(I)$ and ℓ_2 the upper end point $\mathrm{uep}(I)$ of I. One computes the minimal Poincaré rank and the Katz lattice through the following procedure.

Procedure *KATZ*. Input A, output (Ak, m, Pk);

(1) Set $B := -{}^tA$ (dual), $k := p$ and $(\ell_1, \ell_2) := (0, p)$ (initializing).

(2) $I := \{\ell_1, \dots, \ell_2\}$.

(3) **if** $k = \mathrm{uep}(I)$ then $u := 1$ and $k := \ell_1$.

else then $u := 0$ and $k := \ell_2 - 1$.

(4) **end if**

(5) $t := [\frac{\ell_1+\ell_2}{2}]$.

(6) $SAT(B, k)$.

(7) **if** *stable* = *YES* then

if $u = 1$ then output: $m := k$, $Ak := -{}^tBs$ and $Pk :={}^t\ Ps^{-1}$.

else then $\ell_1 := t$ and $\ell_2 := k$

go to (3).

end if

(8) **end if**

(9) **if** *stable* := *NO* then

if $u = 0$ then output: $m := \ell_2$ and $Ak := -{}^tB$.

else $u = 1$ then $\ell_1 := k + 1$ and $\ell_2 := t$

go to (3).

end if

end if

This procedure tests the middle number between the current Poincaré rank (denoted with ℓ_2) and the minimal for which we know *SAT* gives the answer *NO* (denoted with ℓ_1). Therefore, either the upper or the lower half of the interval is discarded according to the result of the test. For large Poincaré ranks, this procedure can avoid a considerable amount of unnecessary computing.

Lemma 5.6. *Let $T \geqslant \frac{n(n-1)}{2}(\mathfrak{p}-\kappa)$. The procedure KATZ applied to the matrix $A_{|T}$ yields a gauge Pk which is a gauge to the Katz lattice attached to the system of matrix A. The matrix $Bk = (A_{|T})_{[Pk]}$ satisfies*

$$Bk - A_{[Pk]} \in z^{T-\frac{n(n-1)}{2}(\mathfrak{p}-\kappa)}\mathrm{M}_n(\mathcal{O}).$$

Proof. Let Λ be the lattice spanned by the canonical basis, ∇ the connection attached to A and $\tilde{\nabla}$ by $B = A_{|T}$. According to Proposition 4.6, we have $\Lambda_K(\nabla) = \Lambda_K(\tilde{\nabla})$, hence P indeed is not an approximate gauge. Lemma 4.7 v) then yields the second statement of the lemma. $\square$

5.3 The Sibuya splitting

Consider the standard notation. Assume that the polar matrix A_{-p} is already split into t blocks according to the distinct eigenvalues $\lambda_1, \ldots, \lambda_t$, i.e

$$A_{-p} = \begin{pmatrix} M_1 & & 0 \\ & \ddots & \\ 0 & & M_t \end{pmatrix} \tag{5.3}$$

where $\mathfrak{Sp}(M_i) \cap \mathfrak{Sp}(M_j) = \emptyset$ if $i \neq j$. For any matrix M denote with (M_{ij}) its block-partitioning according to the matrix A_{-p}. We define the two following linear subspaces $\mathcal{I}$ and $\mathcal{J}$ of $\mathrm{M}_n(\mathbb{C})$

$$\mathcal{I} = \{M \in \mathrm{M}_n(\mathbb{C}) \mid M_{ij} = 0 \text{ if } i \neq j\}$$

and

$$\mathcal{J} = \{M \in \mathrm{M}_n(\mathbb{C}) \mid M_{ii} = 0\}.$$

Obviously, we have

$$\mathrm{M}_n(\mathbb{C}) = \mathcal{I} \oplus \mathcal{J}.$$

Let us write $A = A_{\mathcal{I}} + A_{\mathcal{J}}$ the unique decomposition according to this direct sum. Denote with $\mathcal{I}_z$ (resp. $\mathcal{J}_z$) the subspace of $\mathrm{M}_n(K)$ consisting of series with coefficients in $\mathcal{I}$ (resp. $\mathcal{J}$). Let us recall the following well-known result of linear algebra.

Lemma 5.7. *Let $A \in \mathrm{M}_n(\mathbb{C})$ and $B \in \mathrm{M}_m(\mathbb{C})$ be two square matrices such that $\mathfrak{Sp}(A) \cap \mathfrak{Sp}(B) = \emptyset$. Then the equation $AX - XB = C$ has a unique solution for any matrix $C \in \mathrm{M}_{n\times m}(\mathbb{C})$.*

Corollary 5.8. *The map*

$$\begin{array}{rcl} \mathrm{ad}(A_{-p}) : \mathcal{J} & \longrightarrow & \mathcal{J} \\ X & \longmapsto & [A_{-p}, X] \end{array}$$

is an isomorphism.

Proof. Consider the block-partition $X = (X_{ij})$ of the matrix X according to the blocks of A_{-p}. Then, one has

$$\mathrm{ad}(A_{-p})(X) = \begin{pmatrix} [M_1, X_{11}] & \cdots & M_1 X_{1t} - X_{1t} M_t \\ \vdots & \ddots & \vdots \\ M_t X_{t1} - X_{t1} M_1 & \cdots & [M_t, X_{tt}] \end{pmatrix}.$$

Since $\mathfrak{Sp}(M_i) \cap \mathfrak{Sp}(M_j) = \emptyset$ if $i \neq j$, the equation $M_i X_{ij} - X_{ij} M_j = 0$ has as unique solution $X_{ij} = 0$, hence $X \in \ker \mathrm{ad}(A_{-p})$ if and only if $X \in \mathcal{I}$. Since $\mathcal{I} \cap \mathcal{J} = (0)$, we are done. □

Lemma 5.9. *Given a block-diagonal form* (5.3) *of the matrix A_{-p}, there exists a* unique *gauge $P \in \mathrm{GL}_n(\mathcal{O})$ such that $P - I \in z\mathrm{M}_n(\mathcal{O}) \cap \mathcal{J}_z$ and*

$$A_{[P]} \in \mathcal{I}_z.$$

We call P the Sibuya gauge *attached to the block-digonal matrix A_{-p}.*

Proof. Consider the system (S). In equation (0), one lets $P_0 = I$ and $B_{-p} = A_{-p}$. For all $q < p$, equation (q) has then the simpler form

$$B_{-p+q} = A_{-p+q} + C_q + [A_{-p}, P_q]$$

where

$$C_q = A_{-p+q-1} P_1 - P_1 B_{-p+q-1} + \cdots + A_{-p+1} P_{q-1} - P_{q-1} B_{-p+1}.$$

Assuming by a small induction, that equations up to $(q-1)$ have been so far solved, which means that C_q has been determined, we have

$$\mathrm{ad}(A_{-p})(P_q) = B_{-p+q} - A_{-p+q} - C_q.$$

Put $(B_{-p+q})_{\mathcal{I}} = (A_{-p+q})_{\mathcal{I}} + (C_q)_{\mathcal{I}}$. Then we have $B_{-p+q} - A_{-p+q} - C_q \in \mathcal{J}$. According to Lemma 5.8, the matrix $P_q = \mathrm{ad}(A_{-p})^{-1}(B_{-p+q} - A_{-p+q} - C_q) \in \mathcal{J}$ is uniquely determined, and by construction, the matrix B_{-p+q} is in $\mathcal{I}$. □

Corollary 5.10. *Let ∇ and $\tilde{\nabla}$ be two connections on V. Let Λ be a lattice in V, and let $\mathfrak{p} = \mathfrak{p}_\Lambda(\nabla)$. Assume that we have $\nabla \underset{T}{\overset{\Lambda}{\sim}} \tilde{\nabla}$ with $T \geqslant -\mathfrak{p}$. Let (e) be a Sibuya basis of Λ for ∇. Then*

i) *there exists a Sibuya basis $(\tilde{e})$ of Λ for $\tilde{\nabla}$ such that $(e) \underset{T+\mathfrak{p}}{\overset{\Lambda}{\sim}} (\tilde{e})$,*

ii) $\nabla_\Lambda \underset{T}{\overset{\Lambda}{\sim}} \tilde{\nabla}_\Lambda$.

Proof. Since $\nabla \underset{T}{\overset{\Lambda}{\sim}} \tilde{\nabla}$, we have $\overline{\nabla}^\Lambda = \overline{\tilde{\nabla}}^\Lambda$. Let $A = \mathrm{Mat}(\nabla_\theta, (e))$ and $\tilde{A} = \mathrm{Mat}(\tilde{\nabla}_\theta, (e))$. Let P be the Sibuya gauge attached to A. By Lemma 4.7 iv), we have $A_{[P]} - \tilde{A}_{[P]} \in z^{T+1}\mathrm{M}_n(\mathcal{O})$. According to Lemma 5.9, there is a unique $\tilde{P}$ such that $\tilde{A}_{[\tilde{P}]} \in \mathcal{I}_z$. However we have $(\tilde{A}_{[P]})_{|T} \in \mathcal{I}_z$. Since $(\tilde{A}_{[P]})_{|T} = \Big((\tilde{A}_{|T})_{[P_{|T+p}]}\Big)_{|T}$, we conclude that $P - \tilde{P} \in z^{T+p+1}\mathrm{M}_n(\mathcal{O})$. Hence, the basis $(\tilde{e})$ obtained by $\tilde{P}$ satisfies $(e) \underset{T+\mathfrak{p}}{\overset{\Lambda}{\sim}} (\tilde{e})$.

Let (e) be a Sibuya basis of Λ for ∇, and $(\tilde{e})$ a Sibuya basis of Λ for $\tilde{\nabla}$ as constructed previously. Since $\overline{\nabla}^\Lambda = \overline{\tilde{\nabla}}^\Lambda$, we have $\mathrm{Mat}(f_\Lambda^\nabla, (e)) = \mathrm{Mat}(f_\Lambda^{\tilde{\nabla}}, (\tilde{e})) = \Delta z^{-\mathfrak{p}}$ where $\Delta \in \mathrm{M}_n(\mathbb{C})$ is a diagonal matrix. Let $P = P_{(e),(\tilde{e})}$ and $Q = P^{-1}$. We have by construction $P - I$ and $Q - I \in z^{T+1+\mathfrak{p}}\mathrm{M}_n(\mathcal{O})$. Accordingly, we get $\mathrm{Mat}(f_\Lambda^{\tilde{\nabla}}, (e)) = P\Delta z^{-\mathfrak{p}} Q$, hence

$$\mathrm{Mat}(f_\Lambda^\nabla, (e)) - \mathrm{Mat}(f_\Lambda^{\tilde{\nabla}}, (e)) \in z^{T+1}\mathrm{M}_n(\mathcal{O}),$$

and therefore $\nabla_\Lambda \underset{T}{\overset{\Lambda}{\sim}} \tilde{\nabla}_\Lambda$. □

Corollary 5.11. *Let A be the matrix of ∇_θ in (e). Let $P \in \mathrm{GL}_n(\mathcal{O})$ be the attached Sibuya gauge of A. Let $\tilde{\nabla}$, $\hat{\nabla}$ and $\check{\nabla}$ be the connections whose matrices in (e) are respectively*

$$\tilde{A} = A_{|T}, \quad \hat{A} = \big((A_{[P]})_{|T}\big)_{[P^{-1}]} \quad \textit{and} \quad \check{A} = \big((A_{|T})_{[P_{|T+p}]}\big)_{[P^{-1}]}.$$

Then we have $\nabla \underset{T}{\overset{\Lambda}{\sim}} \tilde{\nabla} \underset{T}{\overset{\Lambda}{\sim}} \hat{\nabla} \underset{T}{\overset{\Lambda}{\sim}} \check{\nabla}$.

We first need a small technical procedure. There exists a constant matrix $P \in \mathrm{GL}_n(\mathcal{O})$ such that

$$P^{-1}A_{-p}P = \begin{pmatrix} M_1 & & 0 \\ & \ddots & \\ 0 & & M_t \end{pmatrix}$$

where $\mathfrak{Sp}(M_i) = \{\lambda_i\}$ such that $\lambda_i \neq \lambda_j$ if $i \neq j$ and moreover M_i is an upper triangular matrix. We call this a *split upper triangular form.*

Procedure *TRS*. Input A, output B, P;

(1) Compute P such that $P^{-1}A_{-p}P$ is a split upper triangular form.

(2) $B := P^{-1}AP$.

If the matrix A already has a block-diagonal structure, we do not want the Sibuya procedure to destroy it. Accordingly, we let the following.

Definition 5.12. Let $g_0 = 1 < g_1 < \cdots < g_s = n$ be a strictly increasing sequence of integers. Let X_{ij} be the block-partition of an $n \times n$ matrix X according to the grid $G = \{(g_i, g_j) \mid 1 \leqslant i, j \leqslant s\}$ of $\{1, \ldots, n\}^2$. We say that X

i) is *grid-scalar* if X is block-diagonal and its diagonal blocks are scalar.

ii) has *grid-simple* eigenvalues, if there exists a grid-scalar matrix D such that $X - D$ is nilpotent.

iii) is *grid-diagonal* if it is block-diagonal according to the block partition induced by G.

For a matrix M we denote with $\operatorname{grid}(M)$ (and we call M-grid) the grid attached to the suitable block-partition attached to the semi-simple part M_s of M. Similarly to the suitable block-partitions, we can also take intersections of grids.

Corollary 5.13. *Given a block-diagonal form* (5.3) *of the matrix A_{-p}, there exists a* unique *A-grid-diagonal Sibuya gauge $P \in \mathrm{GL}_n(\mathcal{O})$ such that $A_{[P]} \in \mathcal{I}_z$ is A-grid-diagonal. We call P the A*-grid Sibuya gauge *attached to the block-diagonal matrix A_{-p}.*

Proof. One simply applies Lemma 5.9 to each block of A. □

Note that Lemma 5.9 may not have been applied in the situation of Corollary 5.13, since nothing prevents differents A-grid-diagonal blocks of A_{-p} to have equal eigenvalues.

Procedure *SIBUYA_SPLIT*. Input A, k, output AS, PS;

(1) (B, P) :=*TRS*(A) and grid := $\operatorname{grid}(B_{-p})$. (Block-diagonalizing of A_{-p}).

(2) $D_{-p} := B_{-p} = \operatorname{diag}(M_1, \ldots, M_t)$.

(3) **for** $m = 1$ to k **do**

(a) $C_m := \sum_{\ell=1}^{i-1} B_{-p+m-\ell} P_\ell - P_\ell D_{-p+m-\ell}$.

for $i = 1$ to n **do**

for $j = 1$ to n **do**

if $i = j$ then $P_{ij}^{(m)} := (0)$

else Solve $M_i P_{ij}^{(m)} - P_{ij}^{(m)} M_j = -C_{ij}^{(m)} - B_{ij}^{(-p+m)}$.

end if

end do

end do

(b) $P_m := (P_{ij}^{(m)})$

(c) $D_{-p+m} :=$ *Grid-diagonal of* B_{-p+m}.

(4) **end do**

(5) Output : $AS := \sum_{m=0}^{k} D_{-p+m} z^{-p+m}$ and $PS := I + \sum_{m=1}^{k} P_m z^m$.

Lemma 5.14. *The procedure SIBUYA_SPLIT with index $T+p$, applied to a matrix $A_{|T}$ yields the matrix $(A_{[P]})_{|T}$ where P is the Sibuya gauge attached to the splitting into blocks of A_{-p}. Moreover, $(A_{[P]})_{|T} = \check{A}_{[P]}$ where the matrix $\check{A}$ satisfies $\check{A} - A \in z^{T+1}\mathrm{M}_n(\mathcal{O})$.*

Proof. The procedure *SIBUYA_SPLIT* with index $T+p$ applied to the matrix $A_{|T}$ yields in fact the matrix $\big((A_{|T})_{[P_{|T+p}]}\big)_{|T}$ where P is the Sibuya gauge attached to A_{-p}. The Sibuya splitting is in $\mathrm{GL}_n(\mathcal{O})$, hence, according to Lemma 4.7 v), we have $(A_{|T})_{[P_{|T+p}]} - A_{[P]} \in z^{T+1}\mathrm{M}_n(\mathcal{O})$. Hence $\big((A_{|T})_{[P_{|T+p}]}\big)_{|T} = (A_{[P]})_{|T}$. The second statement follows from Corollary 5.11. □

5.4 Algorithm LEVELT

Proposition 5.15. *Let ∇ and $\tilde{\nabla}$ be two connections on V. Let Λ be a lattice in V and $\mathfrak{p} = \mathfrak{p}_\Lambda(\nabla)$. If $\nabla \underset{T}{\overset{\Lambda}{\sim}} \tilde{\nabla}$ holds with $T \geqslant \frac{n(n-1)}{2}\mathfrak{p}$ then we have $\Lambda_L(\nabla) = \Lambda_L(\tilde{\nabla})$.*

Proof. Consider with the notation of the proof of Corollary 3.27, the following decreasing chains of sublattices

$$\begin{aligned}\Lambda_0 = \Lambda \supset \Lambda_1 = \Lambda_K(\nabla) \supset \Lambda_2 \\ = (\Lambda_1)_K(\nabla_1) \supset \cdots \supset \Lambda_\kappa = (\Lambda_{\kappa-1})_K(\nabla_{\kappa-1}) = \Lambda_L(\nabla)\end{aligned}$$

where $\kappa = \kappa(\nabla)$ and

$$\begin{aligned}\Lambda_0 = \Lambda \supset \tilde{\Lambda}_1 = \Lambda_K(\tilde{\nabla}) \supset \tilde{\Lambda}_2 \\ = (\tilde{\Lambda}_1)_K(\tilde{\nabla}_1) \supset \cdots \supset \tilde{\Lambda}_{\tilde{\kappa}} = (\tilde{\Lambda}_{\tilde{\kappa}-1})_K(\tilde{\nabla}_{\tilde{\kappa}-1}) = \Lambda_L(\tilde{\nabla})\end{aligned}$$

with $\tilde{\kappa} = \kappa(\tilde{\nabla})$. According to proposition 4.6, since $T \geqslant \frac{n(n-1)}{2}(\mathfrak{p} - \kappa)$, we have $\Lambda_K(\nabla) = \Lambda_K(\tilde{\nabla})$ and therefore we get

$$\Lambda_1 = \tilde{\Lambda}_1 \text{ and } \nabla \underset{T_1}{\overset{\Lambda_1}{\sim}} \tilde{\nabla}$$

with $T_1 = T - \frac{n(n-1)}{2}(\mathfrak{p} - \kappa) \geqslant \frac{n(n-1)}{2}\mathfrak{p} - \frac{n(n-1)}{2}(\mathfrak{p} - \kappa) = \frac{n(n-1)}{2}\kappa$. We have $\nabla_2 = \nabla_{\Lambda_1}$ and $\tilde{\nabla}_2 = \tilde{\nabla}_{\Lambda_1}$. Corollary 5.10 ensures us then that $\nabla_2 \underset{T_1}{\overset{\Lambda_1}{\sim}} \tilde{\nabla}_2$ and we have $T_1 \geqslant \frac{n(n-1)}{2}\kappa \geqslant \frac{n(n-1)}{2}(\kappa - \kappa_1)$ where $\kappa_1 = \kappa(\nabla_1)$.

Let $k < n-1$. Assume that $\Lambda_k = \tilde{\Lambda}_k$ and $\nabla_k \underset{T_k}{\overset{\Lambda_k}{\sim}} \tilde{\nabla}_k$ with $T_k \geqslant \frac{n(n-1)}{2}(\kappa_{k-1} - \kappa_k)$. We have $(\Lambda_k)_K(\nabla_k) = (\Lambda_k)_K(\tilde{\nabla}_k)$, hence $\Lambda_{k+1} = \tilde{\Lambda}_{k+1}$ and $\nabla_{k+1} \underset{T_{k+1}}{\overset{\Lambda_{k+1}}{\sim}} \tilde{\nabla}_{k+1}$ where $T_{k+1} = T_k - \frac{n(n-1)}{2}(\kappa_k - \kappa_{k+1})$. Accordingly, we get

$$T_{k+1} \geqslant T - \sum_{i=0}^{k-1} \frac{n(n-1)}{2}(\kappa_i - \kappa_{i+1}) = T - \frac{n(n-1)}{2}(\mathfrak{p} - \kappa_k) = \frac{n(n-1)}{2}\kappa_k.$$

Hence $T_{k+1} \geqslant \frac{n(n-1)}{2}(\kappa_k - \kappa_{k+1})$ and both $\Lambda_{k+1} = \tilde{\Lambda}_{k+1}$ and $\nabla_{k+1} \underset{T_{k+1}}{\overset{\Lambda_{k+1}}{\sim}} \tilde{\nabla}_{k+1}$ hold. This ends the proof. In particular, we have $\kappa(\nabla) = \kappa(\tilde{\nabla})$. □

Remark 5.16. This result could be improved if we knew a better bound for $\mathfrak{p}_{\mathcal{F}^k}(z^\ell \nabla)$ than the one we give in the proof of proposition 4.6.

We are ready to give the complete algorithm in the non-ramified case.

Procedure *LEVELT*. Input A, output AL, PL, U;

(1) Set $P := I$, grid $:= [1, n]$, $U := (0)$ and $B := A$.

(2) $p := \max(0, -v(B))$.

(3) **if** $p = 0$ then output: $AL := B + U$ and $PL := P$.

(4) **end if**

(5) $(B, T) :=$*TRS*(B).

(6) $P := PT$.

(7) **if** *nilpotent*$(B_{-p}) =$ *TRUE* then $(Bk, Pk) :=$ *KATZ*(B).

$B := Bk$ and $P := P.Pk$

go to (2).

(8) **end if**

(9) **if** *grid-simple*$(B_{-p}) =$ *TRUE* then $B := B - z^{-p}\,\mathrm{diag}(B_{-p})$

$U := U + z^{-p}\,\mathrm{diag}(B_{-p})$.

go to (7).

(10) **end if**

(11) **if** *grid-simple*$(B_{-p}) =$ *FALSE* then $(BS, PS) :=$ *SIBUYA_SPLIT*(B)

$B := BS$ and $P := P.PS$

grid := grid $\wedge$ grid(B_{-p})

go to (9).

(12) **end if**

Proposition 5.17. *The procedure LEVELT applied to the matrix $A_{|T}$ where $T = \frac{n(n-1)}{2}p$ yields a gauge PL to the Levelt lattice of the original lattice, and a matrix AL such that*

$$A_{[PL]} - AL \in z\mathrm{M}_n(\mathcal{O}).$$

Moreover, the exponents attached to A are the eigenvalues of $(AL - U)_0$.

Proof. Let $\tilde{\nabla}$ be the connection whose matrix in the canonical basis is $A_{|T}$. The procedure *LEVELT* is just the transcription of the algorithm described at Section 3.4, which yields $\nabla_\kappa = \nabla^r$, and $\varphi = \sum_{i=0}^{\kappa} f_i$. According to proposition 5.15, we have $\Lambda_L(\nabla) = \Lambda_L(\tilde{\nabla}) = \Lambda_L$ and $\nabla_\kappa \overset{\Lambda_L}{\underset{0}{\sim}} \tilde{\nabla}_\kappa$. The matrix U is then equal to the irregular part AL_{irr} of AL, and it commutes with $(AL - U)_0$. Proposition 4.14 implies that the eigenvalues of $(AL - U)_0$ are then the exponents of $A_{|T}$, hence of A. □

5.5 Ramification

As we previously remarked, ramifying the variable is often needed. Let us recall what happens in that case.

Let $m \in \mathbb{N}$ be the ramification order of ∇. Let $H = K[T]/(T^m - z)$ be the minimal ramified extension, $\mathcal{O}_H$ be the corresponding valuation ring, $\Omega_H = \Omega \otimes_{\mathcal{O}} \mathcal{O}_H$ the attached module of 1-forms, $V_H = V \otimes_K H$ the vector space obtained by extension of scalars, and ∇_H the canonical extension of ∇ to V_H. Consider the canonical decomposition $\nabla = \nabla^r + \omega$ of ∇, and $V_H = \bigoplus_{i=1}^{s} V_i$ the corresponding direct sum.

Let Λ be a lattice in V, and $\Lambda_H = \Lambda \otimes_{\mathcal{O}} \mathcal{O}_H$. Further consider ζ an m-th root of z, and denote with θ_ζ the derivative $\zeta \frac{d}{d\zeta}$.

If $\tilde{\nabla}$ is a regular connection over V_H, and M is a lattice in V_H stable under the action of $\tilde{\nabla}$, we call *compatible residue of $\tilde{\nabla}$ on M* the map $\mathrm{Res}^c_M \tilde{\nabla} : M/\zeta M \longrightarrow M/\zeta M$ obtained by composing ∇ with the residue map $\mathrm{Res}^c : \Omega_H \longrightarrow \mathbb{C}$ that maps $\frac{dz}{z}$ to 1 (and not to m).

Definition 5.18. With the previous notation, define the following.

i) The *Levelt lattice of ∇ with respect to the lattice* Λ is the largest compatible sublattice Λ_L of Λ_H.

ii) The *exponents of ∇ on the lattice* Λ are the eigenvalues $(e_i^\Lambda(\nabla))_{i=1,\dots,n}$ of the compatible residue $\mathrm{Res}^c_{\Lambda_L}(\nabla^r)_H$ of the regular connection $(\nabla^r)_H$ attached to ∇_H with respect to Λ_L.

Over the field extension H of K, we can rewrite the system with respect to the "ramified" variable ζ such that $\zeta^m = z$ by letting $\hat{Y}(\zeta) = Y(\zeta^m)$, which changes (1.1) into

$$\zeta \frac{d}{d\zeta}\hat{Y} = mA(\zeta^m)\hat{Y}. \tag{5.4}$$

Lemma 5.19. *The knowledge of the first k coefficients of the matrix* $A = \mathrm{Mat}(\nabla_\theta, (e))$ *of the connection* ∇_θ *on* V *fully determines the knowledge of the first* mk *coefficients of the matrix* $\tilde{A} = \mathrm{Mat}((\nabla_H)_{\theta_\zeta}, (e \otimes 1))$ *of the extended connection* $(\nabla_H)_{\theta_\zeta}$ *on* V_H.

Proof. It is obvious from equation (5.4) since one has $\tilde{A} = mA(\zeta^m)$. □

Hence replacing $A(z)$ with $B(z) = mA(z^m)$ allows us to assume that the system is unramified.

Corollary 5.20. *The exponents* $e_1, \ldots, e_n$ *given by the algorithm LEVELT applied to* $\tilde{A} = mA(\zeta^m)$ *satisfy the relation*

$$e_i = me_i^{\Lambda}(\nabla) \quad \textit{for } i = 1, \ldots, n.$$

Whereas ramifying the original system up to the order $m = \mathrm{lcm}(1, \ldots, n)$ allows, from a theoretical standpoint to evaluate the number of coefficients of A which are needed to compute the exponents, it is best avoided from a computational point of view, since it increases dramatically the complexity of calculations. Our algorithm presently does not control this feature.

The most likely improvement would be to ramify the variable at the latest possible moment (that is when the polar map on the Katz lattice remains nilpotent), and up to the lowest possible order, for which we can refer to [Ba-C-Lo] in the present volume or [Pf].

Another, less likely, but challenging, improvement would be to compute the determinant map without diagonalizing it. This would allow to compute the determinant factors *and* the ramification order at one shot, without previous ramification, and moreover by a simple computation of a characteristic polynomial with polynomial coefficients.

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Sur les représentations de solutions formelles d'équations différentielles

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Résumé. D'après une conjecture fondamentale de C. Sabbah [11, Chap. I, Conj. 2.5.1.], la classification locale des connexions méromorphes en dimension $n \geq 2$ repose sur l'étude d'une certaine classe de systèmes différentiels quadratiques et intégrables

$$z^k du^k = \sum_{|L_1+\cdots+L_d|\leq 2} u^L \omega_L^k, \quad k = 1, \ldots, d, \tag{F}$$

où les $z^k(x) = x_1^{p_{k1}} \ldots x_n^{p_{kn}}$ sont responsables pour les singularités irrégulières le long des composantes du diviseur à croisements normaux $x_1 \ldots x_n = 0$. Dans cette note, nous étudions les propriétés des solutions formelles $u^k(\hat{x}) \in \mathbb{C}[[\hat{x}]]$ des systèmes

$$z^k X(u^k) = \Phi^k(x, u), \quad k = 1, \ldots, d, \tag{F_X}$$

qui s'obtiennent de (F) le long d'un champ logarithmique X, cf. [4].

Lorsque $\zeta = (\zeta^1, \ldots, \zeta^d)$ et $z(x) = (z^1(x), \ldots, z^d(x))$, nous appelons régulières les séries de la forme $(z^*\varphi)(\hat{x}) \in \mathbb{C}[[\hat{x}]]$, pullback d'une série $\varphi(x, \zeta) \in \mathbb{C}\{x\}[[\zeta]]$.

Le résultat principal que nous démontrons est alors le suivant : si dans (F_X) les coefficients $\Phi_L^k(x)$ sont réguliers, si $\Phi^k(0, 0) = 0$, $k = 1, \ldots, d$ et si la matrice $(\partial\Phi^k/\partial u_l)(0, 0)$ est inversible, alors l'unique solution formelle $u^k(\hat{x})$ telle que $u^k(0) = 0$, $k = 1, \ldots, d$ existe et est régulière.

Comme application, nous regardons des systèmes différentiels intégrables de la forme

$$x^i \frac{\partial u}{\partial x^i} + x^i \frac{\partial \Lambda}{\partial x^i} u = f^i(x), \quad i = 1, \ldots, n, \tag{L}$$

où $f^i(x) \in \mathbb{C}\{x\}$ sont des germes de fonctions holomorphes et $\Lambda(x) = \lambda(x)/x^P$ avec $P = (1, \ldots, 1)$, $\lambda(x) \in \mathbb{C}\{x\}$ et $\lambda(0) \neq 0$. Il s'agit d'une classe très particulière des systèmes considérés dans [11] et nous prouvons que l'unique solution formelle de (L), $u(\hat{x}) \in \mathbb{C}[[\hat{x}]]$ telle que $u(0) = 0$, existe, est régulière et, même, sommable au sens de [10].

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1 Motivation

La motivation pour les résultats présentés dans cette note provient du programme établi par divers auteurs pour l'étude des connexions méromorphes à plusieurs variables complexes (voir [12], [6], [1], [4]), au voisinage d'un point singulier irrégulier. Il se trouve que l'étude des connexions méromorphes est un problème intrinsèquement global qui requiert une connaissance parfaite des outils et résultats sur les faisceaux de D-modules (voir [12]). Pour cette raison, nous ne pouvons ici présenter que quelques aspects locaux.

Soit $x = (x^1, \ldots, x^n)$ et $w = (w^1, \ldots, w^r)$ les coordonnées des espaces affines complexes de dimensions n et r respectivement. La donnée d'une connexion méromorphe se ramène, au voisinage de l'origine, au choix d'un système linéaire intégrable d'équations aux dérivées partielles à coefficients méromorphes,

$$dw^\mu = \sum_{\nu=1}^{r} \alpha^\mu_\nu w^\nu, \quad \alpha^\mu_\nu = \sum_{i=1}^{n} \Gamma^\mu_{i\nu}(x) \frac{dx^i}{x^i} \tag{M}$$

où $\mu = 1, \ldots, r$ et les coefficients $\Gamma^\mu_{i\nu}(x)$ sont des quotients de fonctions holomorphes au voisinage de l'origine. Les conditions d'intégrabilité sont

$$d\alpha^\mu_\nu = \sum_{\rho=1}^{r} \alpha^\mu_\rho \wedge \alpha^\rho_\nu \tag{MCI}$$

où $\mu, \nu = 1, \ldots, r$.

Deux connexions définies par des matrices de formes différentielles α^μ_ν, β^μ_ν sont localement équivalentes lorsque l'on peut faire un changement linéaire inversible $w^\mu(x) = \sum_\rho u^\mu_\rho(x) v^\rho(x)$ avec la propriété suivante

$$du^\mu_\nu = \sum_{\rho=1}^{r} \alpha^\mu_\rho u^\rho_\nu - u^\mu_\rho \beta^\rho_\nu$$

pour tous $\mu, \nu = 1, \ldots, r$. L'objectif est alors d'utiliser de telles transformations pour annuler certains coefficients de β^μ_ν, ce qui correspond à une décomposition en blocs de la connexion méromorphe. En linéarisant le problème, introduisant $u = (u^1, \ldots, u^d) = (u^\mu_\nu)_{\mu,\nu=1,\ldots,r}$ et utilisant les cofacteurs, on se ramène à une classe de problèmes différentiels quadratiques de la forme

$$\begin{aligned} x^i \frac{\partial u^k}{\partial x^i} &= a^k_i(x, u) \\ &= a^k_{i0}(x) + \sum_{l=1}^{d} a^k_{il}(x) u^l + \sum_{|L_1+\cdots+L_d|=2} a^k_{iL}(x) u^L \end{aligned} \tag{D}$$

avec $i = 1, \ldots, n$, $k = 1, \ldots, d$ et $u^L = u_1^{L_1} \ldots u_d^{L_d}$. Ici, les coefficients a^k_{iL} sont fonctions des $\Gamma^\mu_{i\nu}$ et les conditions d'intégrabilité (MCI) entraînent des conditions d'intégrabilité pour (D) de la forme

$$x^j \frac{\partial a_i^k}{\partial x^j} + \sum_{l=1}^{d} a_j^l \frac{\partial a_i^k}{\partial u^l} = x^i \frac{\partial a_j^k}{\partial x^j} + \sum_{l=1}^{d} a_i^l \frac{\partial a_j^k}{\partial u^l} \qquad (DCI)$$

avec $i, j = 1, \dots, n, k = 1, \dots, d$.

Il apparaît, donc, que la situation à plusieurs variables complexes est plutôt comparable à la situation en une variable complexe, où les systèmes quadratiques de la forme (D) jouent un rôle essentiel (voir [2], [8]). Dans le cas de plusieurs variables complexes, il est cependant nécessaire et essentiel de rajouter les conditions d'intégrabilité (DCI).

L'un des aspects fondamentaux du programme pour l'étude des connexions méromorphes est une conjecture de C. Sabbah concernant l'existence de décompositions formelles en blocs. Nous ne souhaitons pas donner l'énoncé précis pour lequel le lecteur pourra consulter [12, Chap. I, Conj. 2.5.1.], mais signalons qu'il conduit naturellement à la question de trouver des solutions formelles pour des systèmes de la forme (D). Dans cette conjecture, il est question de certains facteurs exponentiels dont l'argument est de la forme $\lambda^k(x)/z^k(x)$, $k = 1, \dots, d$ où $\lambda^k(x) \in \mathbb{C}\{x\}$ sont des germes de fonctions holomorphes et $z^k(x) = x^{P_k} = x_1^{P_{k1}} \dots x_n^{P_{kn}}$ définissent des composantes avec multiplicités du diviseur à croisements normaux $x_1 \dots x_n = 0$. Ces facteurs exponentiels introduisent dans (D) des termes linéaires de la forme

$$d\frac{\lambda^k}{z^k}u^k = \frac{1}{z^k}[d\lambda^k - \lambda^k \frac{dz^k}{z^k}]u^k.$$

Le lecteur remarquera que dans cette expression, la forme entre crochets est à coefficients dans $\mathbb{C}\{x\}$ et les pôles sont concentrés dans le facteur $1/z^k$.

En admettant cette conjecture, on est donc naturellement amené à considérer une sous-classe de systèmes de la forme (D), intégrables au sens (DCI). Il s'agit des systèmes définis par des formes différentielles

$$\eta^k = z^k du^k - \sum_j f_j^k(x, u) \frac{dx^j}{x^j}, \qquad k = 1, \dots, d, \qquad (S)$$

où, cette fois ci les coefficients $f_j^k(x, u) \in \mathbb{C}\{x, u\}$ ne présentent plus de pôles, grâce à la présence des facteurs z^k devant du^k. Nous laissons au lecteur le soin d'écrire les conditions d'intégrabilité correspondantes, qui s'obtiennent en développant les identités suivantes

$$\eta^1 \wedge \dots \wedge \eta^d \wedge z^k d\eta^k = 0, \quad k = 1, \dots, d. \qquad (SCI)$$

Nous nous intéressons, alors, à l'étude formel de tels systèmes. Il est commode d'introduire des coordonnées formelles $\hat{x} = (\hat{x}^1, \dots, \hat{x}^n)$ dans lesquelles s'expriment les solutions formelles $u^k(\hat{x}) \in \mathbb{C}[[\hat{x}]]$ du système (S). Cette notation permet de faire la différence entre les séries divergentes $\varphi(\hat{x}) \in \mathbb{C}[[\hat{x}]]$ et les séries formelles

$f(\hat{x}) \in \mathbb{C}[[\hat{x}]]$ convergentes, pour lesquelles l'on dispose de la somme $f(x) \in \mathbb{C}\{x\}$. Dans cette notation, $\hat{x} \mapsto x$ correspond à la possibilité de resommer un objet formel et $x \mapsto \hat{x}$ correspond à la prise du développement de Taylor pour un objet convergent.

Or, la condition d'intégrabilité impose des contraintes importantes aux solutions formelles de tels systèmes. Le lecteur en trouvera un récit détaillé dans [4]. Nous nous limiterons à indiquer que, grâce aux conditions d'intégrabilité, il est possible de se ramener au cas d'un système d'équations différentielles le long d'un champ logarithmique convenablement choisi $X = \sum_j c^j(x)x^j \partial/\partial x^j$. Dans ce cas, on pose $\Phi^k(x,u) = \sum_j c^j(x) f_j^k(x,u)$ et il s'agit, alors, de trouver des solutions du système (que l'on souhaiterait appeler d'équations différentielles ordinaires)

$$z^k X(u^k) = \Phi^k(x,u), \quad k = 1, \ldots, d. \qquad (S_X)$$

C'est cette classe de systèmes qui nous intéresse dans cette note.

2 Introduction

Nous commençons par introduire quelques notations, ce qui nous permet de définir la propriété de stabilité différentielle et de présenter la notion de série formelle régulière. Après une brève présentation du résultat principal, nous donnons le plan de la démonstration.

2.1 Notations

$x = (x^1, \ldots, x^n)$ et $u = (u^1, \ldots, u^d)$ désignent des coordonnées dans les espaces affines complexes de dimensions n et d, respectivement. À l'origine de $\mathbb{C}^n$, nous avons également un système de coordonnées formelles $\hat{x} = (\hat{x}^1, \ldots, \hat{x}^n)$ qui permet d'associer à tout germe holomorphe $\varphi(x,u) \in \mathbb{C}\{x,u\}$ la série formelle correspondante que nous désignerons par $\varphi(\hat{x},u) \in \mathbb{C}\{u\}[[\hat{x}]]$.

Sauf indication contraire, les indices i, j, o, p, q parcourent $1, \ldots, n$ et les indices k, l, r parcourent $1, \ldots, d$, ce qui a pour but d'alléger les expressions contenant des sommes. Malgré cette convention, nous rendrons parfois explicites les limites de certaines sommes lorsque plus de clarté est souhaitable.

Les champs de vecteurs canoniques relativement aux coordonnées x^i seront désignés par ∂_i et les formes duaux par dx^i. Les champs logarithmiques canoniques $x^j\partial_j$ seront abrégés par θ_j et les formes logarithmiques correspondantes dx^j/x^j par ξ^j.

L'action des champs logarithmiques $\theta_j = x^j\partial_j$ sur une série $g(x)$ sera abrégée $g_{,j} = \theta_j(g) = x^j\partial_j(g)$ dans le souci d'éviter des formules trop lourdes dans les démonstrations. Pour les mêmes raisons, l'action de $\partial/\partial u^l$ sur $f(u)$ sera simplifiée en écrivant $f_{;l}$ à la place de $\partial f/\partial u^l$.

Nous désignerons par $m \subset \mathbb{C}\{x\}$ l'idéal maximal de l'anneau local des séries convergentes et par $\hat{m} \subset \mathbb{C}[[\hat{x}]]$ l'idéal maximal de l'anneau local des séries formelles.

2.2 La propriété de stabilité différentielle

Dans toute la suite nous nous fixons des séries convergentes $z(x) = \big(z^1(x), \ldots, z^d(x)\big)$ dans l'idéal maximal $m \subset \mathbb{C}\{x\}$. Nous nous fixons également un champ de vecteurs logarithmique $X = \sum_j c^j(x)\theta_j$ dont les coefficients sont des séries convergentes $c^j(x) \in \mathbb{C}\{x\}$. La propriété suivante sera toujours supposée.

Définition 2.1. Les données $z(x)$ satisfont la propriété de stabilité différentielle suivante, pour tout $k = 1, \ldots, d$,

$$dz^k = z^k\gamma^k, \quad \gamma^k = \sum_{j=1}^{n} \gamma_j^k(x)\frac{dx^j}{x^j}, \quad \gamma_j^k(x) \in \mathbb{C}\{x\}, \qquad (\partial)$$

i.e. avec des coefficients convergents. En particulier, pour tout champ logarithmique comme ci-dessus,

$$\frac{1}{z^k}X(z^k) \in \mathbb{C}\{x\}$$

est une série convergente.

La propriété (∂) est clairement vérifiée lorsque, par exemple, $z^k(x) = u^k(x)x^{P_k}$ pour des $P_k = (P_{k1}, \ldots, P_{kn}) \in \mathbb{N}^n$ et $u^k(x) \notin m$ sont des séries convergentes inversibles (i.e. avec $u^k(0) \neq 0$).

2.3 La notion de série régulière

Soit $\hat{\zeta} = (\hat{\zeta}_1, \ldots, \hat{\zeta}_d)$ un système de coordonnées formelles dans $\mathbb{C}[[\hat{\zeta}]]$. Pour chaque $L = (l_1, \ldots, l_d) \in \mathbb{N}^d$, nous désignerons par $\hat{\zeta}^L$ le monôme $\hat{\zeta}_1^{l_1} \ldots \hat{\zeta}_d^{l_d}$ de degré $|L| = l_1 + \cdots + l_d$.

Si $f(x, \hat{\zeta}) \in \mathbb{C}\{x\}[[\hat{\zeta}]]$ est une série formelle à coefficients convergents, i.e.

$$f(x, \hat{\zeta}) = \sum_{\mu=0}^{+\infty} \sum_{|L|=\mu} f_L(x)\hat{\zeta}^L, \quad f_L(x) \in \mathbb{C}\{x\},$$

alors le pullback z^*f est une série formelle bien définie

$$z^*f(\hat{x}) = \sum_{\mu=0}^{+\infty} \sum_{|L|=\mu} f_L(\hat{x})z^L(\hat{x}) \in \mathbb{C}[[\hat{x}]]$$

puisque $z^L(x) \in m^{|L|}$, ce qui rend la série convergente au sens ultramétrique.

Dans le cas d'un choix général de $z(x)$, de telles séries formelles obtenues par pullback de séries à coefficients convergents sont exceptionnelles et nous les appellerons séries régulières. La dépendance de cette définition en $z(x)$ sera sous-entendue.

De même, lorsque l'on se donne une série formelle $\Phi(\hat{x}, u) \in \mathbb{C}\{u\}[[\hat{x}]]$, nous dirons qu'elle est régulière lorsque il existe $G(x, u, \hat{\zeta}) \in \mathbb{C}\{x, u\}[[\hat{\zeta}]]$ telle que $\Phi = z^*G$.

2.4 Présentation du résultat central

Considérons un système d'équations différentielles le long d'un champ logarithmique X,

$$z^k X(u^k) = \Phi^k(x, u), \quad k = 1, \dots, d, \tag{S_X}$$

où $\Phi^k(x, u)$ sont, par exemple, des polynômes donnés. Si l'on suppose que

- $\Phi^k(0, 0) = 0, k = 1, \dots, d$;
- la matrice $\dfrac{\partial \Phi^k}{\partial u^l}(0, 0) = A_l^k$ est inversible,

alors les preuves utilisées dans [4] s'adaptent pour montrer l'existence d'une et une seule solution formelle $w^k(\hat{x}) \in \mathbb{C}[[\hat{x}]]$ satisfaisant $w^k(0) = 0, k = 1, \dots, d$. Malheureusement, une preuve suivant cette ligne échoue dans la compréhension de l'influence des singularités introduites par les facteurs z^k.

Dans cette note, nous proposons une approche plus générale, dont la conséquence principale est le résultat suivant.

Théorème A. *Soit $z(x) = (z^1(x), \dots, z^d(x))$, avec $z^j(x) \in m \subset \mathbb{C}\{x\}$ où m désigne l'idéal maximal de $\mathbb{C}\{x\}$. Soit $X = \sum_{j=1}^n c^j(x) x^j \partial_j$ un champ de vecteurs logarithmique avec des coefficients $c^j(x) \in \mathbb{C}\{x\}$ et satisfaisant (∂). Enfin, soit $\Phi^k(\hat{x}, u)$, $k = 1, \dots, d$ des séries régulières (relativement à $z(x)$) données, et supposons les faits suivants :*

- *$\Phi^k(0, 0) = 0$;*
- *la matrice $\dfrac{\partial \Phi^k}{\partial u^l}(0, 0) = A_l^k$ est inversible.*

Alors,

1. *Il existe des séries régulières $w^k(\hat{x})$, telles que $w^k(0) = 0$ et*
$$z^k X(w^k) = \Phi^k(\hat{x}, w(\hat{x})), \quad k = 1, \dots, d. \tag{S_X}$$
2. *Soit $w^k(\hat{x}) \in \mathbb{C}[[\hat{x}]]$ des séries formelles telles que $w^k(0) = 0$. Si les équations (S_X) sont satisfaites, alors ces séries formelles sont régulières et, de plus, uniques.*

2.5 Plan de la démonstration

La démonstration sera faite en plusieurs étapes.

- Donner un sens précis aux séries formelles $f(\hat{x}) \in \mathbb{C}[[\hat{x}]]$ admettant une représentation $\sum_{\nu \geq 0} \sum_{|L|=\nu} f_L(x) z^L(x)$.
- Vérifier certaines propriétés élémentaires de ces séries.
- Prouver les analogues des résultats classiques pour des fonctions implicitement définies par $g^k(\hat{x}, w(x)) = 0$ avec des séries régulières $g^k(\hat{x}, u)$.
- Prouver que si le terme indépendant $g^k(\hat{x}, 0)$ dans une telle équation de définition implicite est dans l'idéal engendré par certains $\kappa^\alpha(x)$, alors la série implicitement définie est également dans cet idéal.

– Prouver les faits principaux en utilisant une adaptation de l'esquisse suivante.

Voici l'esquisse simplifiée d'une preuve d'un résultat bien connu, dans le cas convergent avec $n = 1, d = 1$.

Soient m l'idéal maximal, X un champ tel que $X(m) \subset m^2$ et $w(\hat{x}) \in \hat{m}^\nu$ une solution de

$$X(w) = \Phi(x, w(\hat{x})).$$

Si $\Phi(x, v(x)) = 0$ est une solution avec $v(0) = 0$, alors $u = v(x) + \Psi(x, u)\Phi(x, u)$ et donc

$$w(\hat{x}) = v(x) + \Psi(x, w(\hat{x}))X(w).$$

Alors $w(\hat{x}) - v(x) \in m^{\nu+1}$ et $w(\hat{x}) - v(x)$ résout

$$X(u) = \Phi(x, v(x) + u) - X(v).$$

Si l'on admet comme hypothèse de récurrence que $\Phi(x, 0) \in m^\nu$, alors, la nouvelle équation a un terme indépendant dans $m^{\nu+1}$. On conclut que les $v_\nu(x)$ obtenus par récurrence tendent vers zéro au sens ultramétrique.

Si l'on admet que $w(\hat{x})$ est une solution formelle avec $w(0) = 0$, alors les $w(\hat{x}) - (v_0(x) + \cdots + v_\mu(x))$ tendent également vers zéro au sens ultramétrique. On conclut que la solution formelle existe $v(\hat{x}) = \sum_{\nu=0}^{+\infty} v_\nu(x)$ et qu'elle est unique.

3 Préliminaires

3.1 Séries régulières

Supposons que l'on se donne $F^k(x, u, \hat{\zeta}) \in \mathbb{C}\{x, u\}[[\hat{\zeta}]]$, où $\zeta = (\zeta^1, \dots, \zeta^d)$. Alors

$$F^k(x, u, \hat{s}\zeta) = \sum_{\nu=0}^{+\infty} \hat{s}^\nu F^k_\nu(x, u, \zeta)$$

a pour coefficients des polynômes homogènes en ζ,

$$F^k_\nu(x, u, \zeta) = \sum_{|L|=\nu} F^k_L(x, u)\zeta^L.$$

Comme les z^k appartiennent à l'idéal maximal m, on déduit $F^k_\nu(x, u, z(x)) \in m^\nu$ et, par conséquent, pour tout $s \in \mathbb{C}$,

$$F^k(x, u, sz(x)) = \sum_{\nu=0}^{+\infty} s^\nu F^k_\nu(x, u, z(x))$$

converge au sens ultramétrique dans $\mathbb{C}\{u\}[[\hat{x}]]$. On désigne sa somme pour $s = 1$ par $z^*F^k(\hat{x}) \in \mathbb{C}[[\hat{x}]]$. Nous utilisons également la notation $f^k(x, u)$ pour désigner les éléments de $\mathbb{C}\{u\}[[\hat{x}]]$ qui sont de la forme z^*F^k avec $F^k = F^k(x, u, \hat{\zeta})$. Par définition, ce sont des séries formelles admettant une représentation

$$f^k(\hat{x}, u) = z^*F^k(\hat{x}, u) = \sum_{\nu=0}^{+\infty} F^k_\nu(x, u, z(x)),$$

où chaque $F^k_\nu(x, u, \zeta) \in \mathbb{C}\{x, u\}[\zeta]$ est homogène de degré ν en ζ. Il est clair que le choix d'un représentant $F^k(x, u, \hat{\zeta})$ n'est pas unique. Le noyau de z^* est l'idéal engendré par les $\zeta^l - z^l(x)$.

Définition 3.1. Nous appelons une série $f = f(\hat{x}, u) \in \mathbb{C}\{u\}[[\hat{x}]]$ régulière lorsqu'il existe $F = F(x, \hat{\zeta}, u) \in \mathbb{C}\{x, u\}[[\hat{\zeta}]]$ telle que $f = z^*F$, autrement dit, il existe des polynômes homogènes de degré $\nu \geq 0$ en $\zeta = (\zeta^1, \dots, \zeta^d)$,

$$F_\nu(x, \zeta, u) = \sum_{|L|=\nu} F_L(x, u)\zeta^L, \quad F_L(x, u) \in \mathbb{C}\{x, u\}$$

tels que l'on a la représentation

$$f(\hat{x}, u) = \sum_{\nu=0}^{+\infty} F_\nu(x, z(x), u)$$

avec convergence au sens ultramétrique dans $\mathbb{C}\{u\}[[\hat{x}]]$. On écrit $f = f(x, z, u)$ pour les éléments de cette algèbre, notée $\mathbb{C}\{x, u\}[[z]]$.

Nous allons ensuite vérifier que les séries régulières forment une algèbre différentielle. En effet, soient $c^j(x, u), b^l(x, u) \in \mathbb{C}\{x, u\}$ et

$$X = \sum_j c^j(x, u)\theta_j + \sum_l b^l(x, u)\partial_l$$

un germe de champ. Nous avons, par la propriété de stabilité différentielle (∂), $X(z^l) = a^l z^l$ avec $a^l(x, u) = \sum_j c^j(x, u)\gamma^l_j(x)$ dans $\mathbb{C}\{x, u\}$. Donc, si $F = F(x, \hat{\zeta}, u)$,

$$X(z^*F) = z^*\left[X(F) + \sum_l a^l(x)\zeta^l \frac{\partial F}{\partial \zeta^l}\right].$$

On désigne, alors,

$$\hat{X} = \sum_j c^j(x, u)\theta_j + \sum_l b^l(x, u)\partial_l + \sum_l a^l(x)\zeta^l \frac{\partial}{\partial \zeta^l}$$

qui vérifie, pour chaque $F \in \mathbb{C}\{x, u\}[[\hat{\zeta}]]$, $X(z^*F) = z^*(\hat{X}(F))$. On conclut que les séries régulières forment une algèbre stable par dérivation par des champs de la forme $X = \sum_j c^j(x, u)\theta_j + \sum_l b^l(x, u)\partial_l$.

3.2 Fonctions régulières implicites

Soient $f^k = f^k(\hat{x}, u)$ régulières. Si $y^k = y^k(\hat{x})$ sont régulières et $y^k(0) = 0$, nous pouvons définir $y^* f^k$. Or, $y^k = z^* Y^k$, $f^k = z^* F^k$ et

$$f^k(\hat{x}, y(\hat{x})) = F^k\big(x, z(x), Y\big(x, z(x)\big)\big).$$

En particulier $g^k = y^* f^k$ vérifie $g^k = z^* G^k$ avec $G^k(x, \zeta) = F^k(x, \zeta, Y(x, \zeta))$.

Lemme 3.2. *Soient $f^k = f^k(\hat{x}, u)$ et $y^k(\hat{x})$ régulières. Supposons que $y^l(0) = 0$, $l = 1, \ldots, d$. Alors les $y^* f^k = f^k(\hat{x}, y(\hat{x}))$ sont régulières.*

Inversement, les séries formelles définies implicitement par des équations régulières avec partie linéaire inversible sont régulières.

Lemme 3.3. *Soit $f^k = f^k(\hat{x}, u)$, $k = 1, \ldots, d$ des séries régulières et supposons les faits suivants :*

- *$f^k(0, 0) = 0$, $k = 1, \ldots, d$;*
- *la matrice $\dfrac{\partial f^k}{\partial u^l}(0, 0) = A^k_l$ est inversible.*

Alors, il existe $y^k = y^k(\hat{x})$ régulières, uniques telles que $y^k(0) = 0$ et $y^* f^k = 0$, i.e. *$f^k(\hat{x}, y(\hat{x})) = 0$.*

Démonstration. Par hypothèse, nous pouvons choisir $F^k = F^k(x, \hat{\zeta}, u)$ tels que $f^k = z^* F^k$. Donc $f^k_{.l} = z^* F^k_{.l}$. Comme $z^l(0) = 0$, $F^k(0, 0, 0) = f^k(0, 0) = 0$ et $F^k_{.l}(0, 0) = f^k_l(0, 0) = A^k_l$.

Par la variante formelle du théorème des fonctions implicites pour les variables (x, ζ), on conclut qu'il existe des $Y^k = Y^k(\hat{x}, \hat{\zeta})$, uniques tels que $Y^k(0, 0) = 0$ et

$$Y^* F^k(\hat{x}, \hat{\zeta}) = F^k(x, \hat{\zeta}, Y(\hat{x}, \hat{\zeta})) = 0.$$

Supposons que $Y^k = Y^k(x, \hat{\zeta})$ a des coefficients convergents. Alors, il suffit de prendre

$$y^k(\hat{x}) = z^* Y^k = Y^k(x, z(x)).$$

Il est alors clair que $y^* f^k = z^* Y^* z^* F^k = z^* Y^* F^k = 0$ comme souhaité.

Il s'agit donc de montrer que $Y^k = Y^k(x, \hat{\zeta})$ a des coefficients convergents. Or,

$$F^k(x, \hat{s}\zeta, u) = F^k(x, \zeta, \hat{s}, u) = \sum_{\nu=0}^{+\infty} \hat{s}^\nu F^k_\nu(x, \zeta, u),$$

$$Y^l(\hat{x}, \hat{s}\zeta) = Y^l(x, \zeta, \hat{s}) = \sum_{\nu=0}^{+\infty} \hat{s}^\nu Y^l_\nu(\hat{x}, \zeta)$$

et

$$F^k(x, \zeta, \hat{s}, Y(\hat{x}, \zeta, \hat{s})) = 0.$$

Quitte à regarder (x, ζ, s) comme de nouvelles coordonnées $(x_1, \dots, x_{n+d}, s)$, nous sommes donc ramenés à montrer le cas particulier suivant de l'énoncé. Si

$$F^k = F^k(x, \hat{s}, u), \quad F^k(0, \hat{s}, 0) = 0, \quad F^k_{.l}(0, 0, 0) = A^k_l$$

et $Y^l = Y^l(\hat{x}, \hat{s})$ est l'unique solution $Y^l(0, 0) = 0$ de $Y^* F^k = 0$, alors $Y^k = Y^k(x, \hat{s})$. Autrement dit, les coefficients Y^k_ν sont en réalité des germes holomorphes. Par l'unicité, il suffit de produire une solution $Y^l(x, \hat{s})$ avec $Y^l(0, 0) = 0$.

Nous allons démontrer par récurrence sur $\mu \geq 0$ qu'il existe

$$_\mu F^k = {_\mu F^k}(x, \hat{s}, u), \quad Y^l_\mu = Y^l_\mu(x)$$

tels que

$$Y^l_\mu(0) = 0, \quad {_\mu F^k}(0, \hat{s}, 0) = 0, \quad {_\mu F^k_{.l}}(0, 0, 0) = A^k_l$$

et

$$_0 F^k(x, \hat{s}, u) = F^k(x, \hat{s}, u), \tag{H_{-1}}$$

$$_\nu F^k(x, \hat{s}, Y_\nu(x) + su) = {_{\nu+1} F^k}(x, \hat{s}, u)s, \quad \nu = 0 \dots \mu. \tag{H_μ}$$

Par définition (H_{-1}) est vérifiée. Soit $G^k(x, u) = {_0 F^k}(x, 0, u)$. Alors, $G^k(0, 0) = 0$ et $G^k_{.l}(0, 0) = A^k_l$. On conclut qu'il existe $Y^k_0(x)$, uniques avec $Y^k_0(0) = 0$ tels que

$$_0 F^k(x, 0, Y_0(x)) = 0$$

ce qui montre que ${_0 F^k}(x, \hat{s}, Y_0(x) + su)$ est divisible par s. Nous pouvons donc résoudre de manière unique (H_0).

Lorsque Y^l_ν, ${_\nu F^k}$, $\nu = 0, \dots, \mu$ ont les propriétés souhaités, et ${_{\mu+1} F^k}$ sont définis satisfaisant (H_μ), nous avons

$$_{\mu+1} F^k_{.l}(x, \hat{s}, u) = {_\mu F^k_{.l}}(x, \hat{s}, Y_\nu(x) + su)$$

et, à fortiori ${_{\nu+1} F^k_{.l}}(0, 0, 0) = {_\nu F^k_{.l}}(0, 0, 0) = A^k_l$. De même,

$$_\mu F^k(0, \hat{s}, 0) = {_{\mu+1} F^k}(0, \hat{s}, 0)s$$

ce qui montre que ${_{\mu+1} F^k}(0, \hat{s}, 0) = 0$. Donc ${_{\mu+1} F^k}$ ont les propriétés souhaitées. Donc, si $G^k(x, u) = {_{\mu+1} F^k}(x, 0, u)$, nous avons $G^k(0, 0) = 0$, $G^k_{.l}(0, 0) = A^k_l$. On en conclut qu'il existe $Y^l_{\mu+1}(x)$, uniques tels que $Y^l_{\mu+1}(0) = 0$ et

$$_{\mu+1} F^k(x, 0, Y_{\mu+1}(x)) = 0$$

ce qui montre que l'on peut résoudre $(H_{\mu+1})$, ce qui achève la démonstration par récurrence.

Comme

$$_\nu F^k(x, \hat{s}, Y_\mu(x) + su)s^\mu = {_{\mu+1} F^k}(x, \hat{s}, u)s^{\mu+1},$$

on vérifie sans difficulté que

$$F^k(x, \hat{s}, Y_0(x) + \cdots + s^\mu Y_\mu(x) + su) = {}_{\mu+1}F^k(x, \hat{s}, u)s^{\mu+1} \qquad (*)$$

Soit

$${}_\mu Y^l(x, \hat{s}) = \sum_{\nu=0}^{+\infty} Y^l_{\mu+\nu}(x)\hat{s}^\nu.$$

Alors,

$${}_0Y^l(x, \hat{s}) = Y^l_0(x) + \cdots + s^\mu Y^l_\mu(x) + ss^\mu \ {}_{\mu+1}Y^l(x, \hat{s})$$

ce qui prouve, avec $u^l = s^\mu \ {}_{\mu+1}Y^l(x, \hat{s})$ dans $(*)$ que

$$F^k(x, \hat{s}, {}_0Y(x, \hat{s})) = {}_{\mu+1}F^k(x, \hat{s}, s^\mu \ {}_{\mu+1}Y(x, \hat{s}))s^{\mu+1}$$

puisque $\mu \geq 0$ est arbitraire, on conclut que ${}_0Y^k = {}_0Y^k(x, \hat{s})$ et ${}_0Y^*F^k = 0$, ce qui achève la démonstration. □

Le résultat suivant nous donne un contrôle sur la fonction implicitement définie en termes du coefficient indépendant de l'équation

Lemme 3.4. *Soient $g^k(\hat{x}, u)$ réguliers, tels que $g^k_{.l}(0, 0) = A^k_l$ soit une matrice inversible et, pour un ensemble I fini,*

$$g^k(\hat{x}, 0) = \sum_{\alpha\in\mathrm{I}} \kappa^\alpha(x)b^k_\alpha(\hat{x})$$

où les $b^k_\alpha(\hat{x})$ sont réguliers, $b^k_\alpha(0) = 0$ et $\kappa^\alpha(x)$ sont des germes qui vérifient $\kappa^\alpha(0) = 0$. Dans ces conditions, il existe d'uniques séries régulières $v^k = v^k(\hat{x})$ telles que $v^k(0) = 0$ et

$$g^k(\hat{x}, v^k(\hat{x})) = 0.$$

De plus, on peut écrire

$$v^k(\hat{x}) = \sum_{\alpha\in\mathrm{I}} \kappa^\alpha(x)a^k_\alpha(\hat{x})$$

avec des quotients $a^k_\alpha(\hat{x})$ réguliers tels que $a^k_\alpha(0) = 0$.

Démonstration. L'unicité résulte de l'unicité formelle. Nous pouvons admettre $b^k_\alpha(\hat{x}) = \sum_j x^j b^k_{\alpha j}(\hat{x})$ avec régularité. Quitte à remplacer I par $\mathrm{I} \times \{1, \ldots, n\}$ et κ^α par $\kappa^{\alpha j}(x) = x^j\kappa^\alpha(x)$, nous pouvons nous ramener à démontrer l'énoncé sans vérifier si $a^k_\alpha(0) = 0$.

Nous pouvons désigner $y = (y^\alpha, \alpha \in \mathrm{I})$ comme de nouvelles variables et

$$f^k(\hat{x}, y, u) = g^k(\hat{x}, u) + \sum_\alpha (y^\alpha - \kappa^\alpha(x))b^k_\alpha(\hat{x}).$$

Le membre de droite est de toute évidence polynomial pour les nouvelles variables avec des coefficients réguliers. On conclut que f^k est régulier par rapport aux variables étendues (x, y). Par conséquent, le lemme 3.3 produit une solution régulière $u^k(\hat{x}, \hat{y})$, unique telle que $u^k(0,0) = 0$ et $f^k(\hat{x}, y, u^k(\hat{x}, \hat{y})) = 0$. Or, $u^k(\hat{x}, 0)$ est une solution régulière vérifiant $u^k(0,0) = 0$ de $f^k(\hat{x}, 0, u) = 0$. Mais $f^k(\hat{x}, 0, 0) = 0$ aussi, donc, par unicité, on conclut $u^k(\hat{x}, 0) = 0$. En particulier, si $u^k = z^* U^k$, on conclut $U^k(x, 0, z(x)) = 0$. Comme

$$U^k(x, y, \hat{\zeta}) - U^k(x, 0, \hat{\zeta}) = \sum_{\alpha \in I} y^\alpha A^k_\alpha(x, y, \hat{\zeta})$$

il en résulte

$$u^k(\hat{x}, y) = \sum_{\alpha \in \mathrm{I}} y^\alpha A^k_\alpha(x, y, z(x))$$

et donc, si l'on pose $v^k = \kappa^* u^k$, $a^k_\alpha(\hat{x}) = A^k_\alpha(x, \kappa(x), z(x))$,

$$v^k(\hat{x}) = u^k(\hat{x}, \kappa(x)) = \sum_{\alpha \in \mathrm{I}} \kappa^\alpha(x) a^k_\alpha(\hat{x})$$

qui vérifient bien $g^k(\hat{x}, v^k(\hat{x})) = 0$. $\square$

4 Existence et unicité

Nous sommes, maintenant, en mesure de prouver le résultat principal de cette note. La démonstration se fait en trois étapes, utilisant des arguments par récurrence. Les idées clefs sont les suivantes.

Lorsque l'on se donne le système abstrait d'équations

$$z^k X(u^k) = \Phi^k(\hat{x}, u), \quad k = 1, \ldots, d, \tag{S_X}$$

il existe des séries convergentes de la forme

$$w^k_\mu(x) = \sum_{|L|=\mu} w^k_L(x) z^L(x) \in \mathbb{C}\{x\}$$

telles que, en faisant le changement de coordonnées $u \mapsto {}_\mu u$ induit par

$$u^k = \sum_{\nu=0}^{\mu-1} w^k_\nu(x) + {}_\mu u^k,$$

nous obtenons un système de la même forme, avec de nouvelles ${}_\mu\Phi^k(\hat{x}, {}_\mu u)$

$$z^k X({}_\mu u^k) = {}_\mu\Phi^k(\hat{x}, {}_\mu u), \quad k = 1, \ldots, d.$$

Cela permet de montrer l'existence d'une solution $W^k(\hat{x})$ de (S_X), régulière et satisfaisant $W^k(0) = 0$, en prenant

$$W^k(\hat{x}) = \sum_{\nu=0}^{+\infty} w_\nu^k(x).$$

La deuxième partie consiste en vérifier que si $w^k(\hat{x})$ définissent une solution formelle de (S_X) avec $w^k(0) = 0$, alors on doit avoir

$$w^k(\hat{x}) = \sum_{\nu=0}^{\mu-1} w_\nu^k(x) + {}_\mu w^k(\hat{x})$$

où les ${}_\mu w^k(\hat{x})$ sont de la forme

$${}_\mu w^k(\hat{x}) = \sum_{|L|=\mu} {}_L w^k(\hat{x}) z^L(x) \in \mathbb{C}[[x]].$$

Cela montre que $w^k(\hat{x})$ est régulière et, en réalité, identique à $W(\hat{x})$, ce qui montre l'unicité.

Théorème 4.1. *Soit $z(x) = (z^1(x), \dots, z^d(x))$, avec $z^j(x) \in m \subset \mathbb{C}\{x\}$ où m désigne l'idéal maximal de $\mathbb{C}\{x\}$. Soit $X = \sum_{j=1}^n c^j(x) x^j \partial_j$ un champ de vecteurs logarithmique avec des coefficients $c^j(x) \in \mathbb{C}\{x\}$ et satisfaisant (∂). Enfin, soit $\Phi^k(\hat{x}, u)$, $k = 1, \dots, d$ des séries régulières (relativement à $z(x)$) données, et supposons les faits suivants :*

- $\Phi^k(0,0) = 0$;
- *la matrice* $\dfrac{\partial \Phi^k}{\partial u^l}(0,0) = A_l^k$ *est inversible.*

Alors,

1. *il existe des séries régulières $w^k(\hat{x})$, telles que $w^k(0) = 0$ et*
$$z^k X(w^k) = \Phi^k(\hat{x}, w(\hat{x})), \quad k = 1, \dots, d; \qquad (S_X)$$
2. *soit $w^k(\hat{x}) \in \mathbb{C}[[\hat{x}]]$ des séries formelles telles que $w^k(0) = 0$. Si les équations (S_X) sont satisfaites, alors ces séries formelles sont régulières et, de plus, uniques.*

Démonstration. La preuve se fait en trois parties. La première partie est un résultat d'existence faisant intervenir uniquement le système d'équations considéré. La deuxième partie est un résultat d'existence faisant intervenir une solution formelle donnée. La troisième partie est un résultat d'unicité en un sens fort pour la solution considérée dans la deuxième partie.

Pour la première partie, nous devons montrer l'existence de certaines familles d'objets $w_L^k(x)$, $w_\mu^k(x)$, ${}_L g^k(\hat{x})$, ${}_\mu g^k(\hat{x})$, ${}_\nu \Phi^k(\hat{x}, u)$ dont la description est la suivante. Pour $\mu \in \mathbb{N}$ donné,

$$w_L^k(x) \in \mathbb{C}\{x\}, \quad 0 \le |L| \le \mu, \ k = 1, \dots, d, \qquad (B_\mu)$$

sont des coefficients convergents définissant des polynômes homogènes de degrés $0 \leq |L| = \nu \leq \mu$, dont l'on obtient par pullback des séries convergentes

$$w_\mu^k(x) = \sum_{|L|=\mu} w_L^k(x) z^L(x) \in \mathbb{C}\{x\}. \tag{B'_μ}$$

$$_L g^k(\hat{x}) \in \mathbb{C}\{x\}[[z]], \quad 0 \leq |L| \leq \nu,\ \nu \leq \mu k = 1, \dots, d, \tag{C_μ}$$

sont des coefficients réguliers définissant des polynômes homogènes de degré $0 \leq |L| = \nu \leq \mu$, dont l'on obtient par pullback des séries régulières

$$_\mu g^k(\hat{x}) = \sum_{|L|=\mu} {}_L g^k(\hat{x}) z^L(x) \in \mathbb{C}\{x\}[[z]]. \tag{C'_μ}$$

Enfin,

$$_\nu \Phi^k(\hat{x}, u) \in \mathbb{C}\{x, u\}[[z]], \quad 0 \leq \nu \leq \mu,\ k = 1, \dots, d, \tag{D_μ}$$

sont des séries régulières.

Nous considérons, maintenant l'hypothèse de récurrence suivante. Pour $\mu \in \mathbb{N}$ donné, les familles (C_μ),(D_μ) sont bien définies et en utilisant (C'_μ),(D_μ), la condition suivante est satisfaite

$$\frac{\partial\, {}_\mu\Phi^k}{\partial u^l}(0,0) = A_l^k, \quad {}_\mu\Phi^k(\hat{x}, 0) = {}_\mu g^k(\hat{x}). \tag{Z_μ}$$

De plus, si $\mu \geq 1$, la famille $(B_{\mu-1})$ est bien définie et en utilisant $(B'_{\mu-1})$, la condition suivante est satisfaite.

$$_\mu\Phi^k(\hat{x}, u) = {}_{\mu-1}\Phi^k(\hat{x}, w_{\mu-1}^k(x) + u) - z^k X(w_{\mu-1}^k). \tag{$R_{\mu-1}$}$$

Considérons d'abord le cas $\mu = 0$. Nous devons construire (C_0),(D_0) et vérifier (Z_0). Par hypothèse $\Phi^k(\hat{x}, u)$ est régulière donc (D_0) s'obtient en prenant ${}_0\Phi^k = \Phi^k$ et (C_0) en prenant ${}_0 g^k(\hat{x}) = \Phi^k(\hat{x}, 0)$. Toujours par hypothèse, la première partie de (Z_0) est satisfaite et la deuxième partie l'est par choix de (C_0).

Supposons donc que l'hypothèse de récurrence soit vraie pour $\mu \geq 0$ donné.

Nous devons d'abord montrer que (Z_μ) permet de bien définir (B_μ).

En effet, le lemme 3.4 appliqué au cas $\mathrm{I} = \{L \in \mathbb{N}^d \mid |L| = \mu\}$, $\kappa^L = z^L$ montre qu'il existe des ${}_L v^k(\hat{x})$ régulières telles que

$$_\mu v^k(\hat{x}) = \sum_{|L|=\mu} z^L(x)\, {}_L v^k(\hat{x})$$

soit l'unique solution avec ${}_\mu v^k(0) = 0$ de ${}_\mu\Phi^k(\hat{x}, u) = 0$. Or, par définition de série régulière, cela implique qu'il existe ${}_L r_m^k(\hat{x})$ régulières et $w_L^k(x)$ convergentes telles que

$$_L v^k(\hat{x}) = w_L^k(x) + \sum_{|M|=1} z^M\, {}_L r_M^k(\hat{x}).$$

D'où l'on déduit

$$ {}_{\mu}v^k(\hat{x}) = \sum_{|L|=\mu} z^L(x) w_L^k(x) + \sum_{|L|=\mu+1} z^L(x)\, {}_L r^k(\hat{x}) $$

avec $w_L^k(x) \in \mathbb{C}\{x\}$ et ${}_L r^k(\hat{x}) \in \mathbb{C}\{x\}[[z]]$. Ces $w_L^k(x)$, $|L| = \mu$ complètent donc $(B_{\mu-1})$ pour définir (B_μ). Il sera utile plus loin de remarquer que si l'on pose

$$ {}_{\mu+1}r^k = \sum_{|L|=\mu+1} z^L(x)\, {}_L r^k(\hat{x}) $$

alors, les $w_\mu^k(x)$ définis par (B'_μ) vérifient

$$ {}_{\mu}v^k(\hat{x}) = w_\mu^k(x) + {}_{\mu+1}r^k(\hat{x}). \tag{$**$} $$

Nous allons, en suite, utiliser (B_μ) et (D_μ) pour définir ${}_{\mu+1}\Phi^k(\hat{x}, u)$ par (R_μ), qui est alors trivialement satisfaite, nous pouvons donc compléter (D_μ) par ${}_{\mu+1}\Phi^k(\hat{x}, u)$ ainsi construite pour définir $(D_{\mu+1})$.

Remarquons par ailleurs que, alors, la première partie de $(Z_{\mu+1})$ résulte de (Z_μ) et de (R_μ).

Il s'agit, enfin, de définir $(C_{\mu+1})$ de manière à satisfaire la deuxième partie de $(Z_{\mu+1})$. Pour ce faire, il est clair que l'on doit simplement calculer ${}_{\mu+1}\Phi^k(\hat{x}, 0)$ et prouver que c'est de la forme $(C'_{\mu+1})$ pour une définition convenable des ${}_L g^k(\hat{x})$, $|L| = \mu + 1$.

Rappelons que dans la construction des solutions $v^k(x)$ on obtient des $\Psi_l^k(\hat{x}, u)$ régulières telles que

$$ {}_{\mu}\Phi^k(\hat{x}, u) = \sum_l (u^l - v^l(x))\Psi_l^k(\hat{x}, u), $$

entraînant

$$ {}_{\mu}\Phi^k(\hat{x}, w_\mu^k(x)) = \sum_{|L|=\mu+1} z^L(x)\Big[-\sum_l {}_L r^l(\hat{x})\Psi_l^k(\hat{x}, w_\mu^k(x))\Big]. \tag{I} $$

Par ailleurs, l'hypothèse de stabilité différentielle (∂) implique que pour tout $R \in \mathbb{N}^d$, le terme entre parenthèses dans

$$ X(z^R) = z^R\Big(\sum_r \frac{R_r}{z^r} X(z^r)\Big), $$

est convergent. On déduit alors de (B'_μ) que

$$ z^k X(w_\mu^k) = \sum_{|R|=\mu} z^{R+e_k}(x)\Big[\Big(X + \sum_r \frac{R_r}{z^r} X(z^r)\Big)(w_R^k)\Big]. \tag{II} $$

En utilisant (R_μ), ceci montre que l'on peut définir les ${}_L g^k(\hat{x})$, $|L| = \mu + 1$ par les termes entre crochets dans (I), (II). Cela permet de définir $(C_{\mu+1})$ partant de (C_μ)

et alors, en utilisant $(C'_{\mu+1})$ et (I),(II), on conclut

$$ {}_{\mu+1}g^k(\hat{x}) = {}_{\mu+1}\Phi^k(\hat{x}, 0) = \sum_{|L|=\mu+1} z^L(x)\, {}_Lg^k(\hat{x}) $$

comme souhaité pour vérifier $(Z_{\mu+1})$.

Nous avons ainsi démontré que l'hypothèse de récurrence est vérifiée pour $\mu + 1$, ce qui montre que (B_μ),(C_μ),(D_μ) sont bien définies pour tout $\mu \in \mathbb{N}$ et, utilisant (B'_μ),(C'_μ), satisfont les conditions (R_μ),(Z_μ).

Maintenant, utilisant $(B'_{\mu+\rho})$, nous avons $w^k_{\mu+\rho} \in m^{\mu+\rho} \subset \mathbb{C}\{x\}$, où m est l'idéal maximal de $\mathbb{C}\{x\}$. Donc, la série

$$ \sum_{\mu=0}^{+\infty} w^k_{\mu+\rho}(x) $$

converge au sens ultramétrique vers une série formelle qui est, par construction régulière, ${}_\rho W^k(\hat{x}) \in \mathbb{C}\{x\}[[z]]$. Plus précisément, ${}_\rho W^k(\hat{x}) \in \hat{m}^\rho \subset \mathbb{C}[[x]]$ où $\hat{m}$ désigne l'idéal maximal de $\mathbb{C}[[x]]$.

Or, en utilisant (R_ρ) pour $\nu \le \rho \le \mu$, on déduit

$$ {}_\nu\Phi^k(\hat{x},\ {}_\nu W^k(\hat{x})) - z^k X({}_\nu W^k) = {}_{\mu+1}\Phi^k(\hat{x},\ {}_{\mu+1}W^k(\hat{x})) - z^k X({}_{\mu+1}W^k) $$

En suite, remarquons que ${}_{\mu+1}\Phi^k(\hat{x}, 0) = {}_{\mu+1}g^k(\hat{x}) \in \hat{m}^{\mu+1}$, entraînant

$$ {}_\nu\Phi^k(\hat{x},\ {}_\nu W^k(\hat{x})) - z^k X({}_\nu W^k) \in \hat{m}^{\mu+1} $$

pour tout $\mu \ge \nu$, ce qui nous permet d'affirmer

$$ z^k X({}_\nu W^k) = {}_\nu\Phi^k(\hat{x},\ {}_\nu W^k(\hat{x})). $$

En particulier, si nous désignons

$$ W^k(\hat{x}) = {}_0W^k(\hat{x}) = \sum_{\mu=0}^{+\infty} w^k_\mu(x), \tag{+} $$

il s'agit de séries régulières satisfaisant $W^k(0) = 0$ et

$$ z^k X(W^k) = \Phi^k(\hat{x}, W(\hat{x})). $$

Nous avons donc conclut la première partie de la preuve, montrant l'existence d'une solution régulière $W^k(\hat{x})$ telle que $W^k(0) = 0$.

La deuxième partie de la preuve consiste en prouver l'existence de certaines séries formelles

$$ {}_Lw^k \in \mathbb{C}[[x]], \quad |L| \le \mu,\ k = 1, \dots, d, \tag{A_μ} $$

définissant certains polynômes homogènes de degrés $|L| = \nu \leq \mu$ dont l'on extrait par pullback les séries formelles

$$ {}_{\mu}w^k(\hat{x}) = \sum_{|L|=\mu} {}_{L}w^k(\hat{x}) z^L(x) \in \mathbb{C}[[x]]. \qquad (A'_{\mu}) $$

Plus précisément, nous faisons l'hypothèse de récurrence suivante, pour $\mu \in \mathbb{N}$.

Les (A_{μ}) sont bien définies et, utilisant (A'_{μ}), satisfont

$$ z^k X({}_{\mu}w^l)(\hat{x}) = {}_{\mu}\Phi^k(\hat{x}, {}_{\mu}w^l(\hat{x})). \qquad (W_{\mu}) $$

De plus, si $\mu \geq 1$, la condition suivante est aussi satisfaite

$$ {}_{\mu-1}w^k(\hat{x}) = w^k_{\mu-1}(x) + {}_{\mu}w^k(\hat{x}). \qquad (S_{\mu-1}) $$

Considérons d'abord le cas $\mu = 0$. Puisque par hypothèse l'on se donne des séries formelles $w^k(\hat{x})$ qui vérifient $w^k(0) = 0$ et $z^k X(w^k) = \Phi^k(\hat{x}, w(\hat{x}))$, il suffit de définir (A_0) par ${}_0w^k = w^k$ pour que (W_0) soit vérifiée.

Supposons donc que l'hypothèse de récurrence soit vérifiée pour $\mu \geq 0$.

Rappelons que les séries $\Psi^k_l(\hat{x}, u)$ construites dans la première partie définissent une matrice $d \times d$ vérifiant $\Psi^k_l(0, 0) = A^k_l$, qui est inversible. Nous pouvons donc construire la matrice inverse de $\Psi^k_l(\hat{x}, u)$, dont les coefficients $G^l_k(\hat{x}, u)$ sont des séries régulières. Nous obtenons, alors, l'expression suivante,

$$ u^l = v^l(\hat{x}) + \sum_k G^l_k(\hat{x}, u)\, {}_{\mu}\Phi^k(\hat{x}, u) $$

et, utilisant $(**)$ aussi bien que (W_{μ}), on déduit que

$$ {}_{\mu}w^k(\hat{x}) = w^k_{\mu}(x) + {}_{\mu}r^k(\hat{x}) + \sum_m z^m G^k_m(\hat{x}, {}_{\mu}w^m) X({}_{\mu}w^m). \qquad (III) $$

Mais, en utilisant la propriété de stabilité différentielle (∂), cela entraîne

$$ X({}_{\mu}w^m) = \sum_{|M|=\mu} z^M(x)\Big(X + \sum_r \frac{1}{z_r} X(z_r^{M_r})\Big)({}_{M}w^m). \qquad (IV) $$

Nous pouvons donc définir, pour chaque $|L| = \mu + 1$, des ${}_{L}w^k(\hat{x}) \in \mathbb{C}[[x]]$ par l'expression suivante, obtenue en regroupant les facteurs de $z^L(x)$ dans l'identité résultant de (III) et (IV),

$$ {}_{L}w^k(\hat{x}) = {}_{L}r^k(\hat{x}) + \sum_{M+e_m=L} G^k_m(\hat{x}, {}_{\mu}w(\hat{x}))(X + \sum_r \frac{X(z_r^{M_r})}{z_r})({}_{M}w^m). $$

Nous obtenons ainsi la famille $(A_{\mu+1})$ par extension de (A_{μ}) et, par construction, (S_{μ}) est vérifiée. Le lecteur vérifiera que la construction est ainsi faite que $(W_{\mu+1})$ est également vérifiée, ce qui achève la preuve par récurrence.

En particulier, en utilisant (S_ρ) pour $\nu \leq \rho \leq \mu$, on conclut

$$ {}_{\nu}w^k(\hat{x}) = \sum_{\rho=\nu}^{\mu} w_\rho^k(x) + {}_{\mu+1}w^k(\hat{x}). \tag{++}$$

D'après $(A'_{\mu+1})$, les restes sont dans une puissance de l'idéal maximal, ${}_{\mu+1}w^k(\hat{x}) \in \hat{m}^{\mu+1}$. En particulier, la série

$$\sum_{\mu=0}^{\infty} w_{\mu+\rho}^k(x)$$

converge au sens ultramétrique vers ${}_{\rho}w^k(\hat{x})$. Donc,

$$w^k(x) = {}_{0}w^k(\hat{x}) = \sum_{\mu=0}^{\infty} w_\mu^k(x) \tag{++}$$

ce qui prouve que les w^k sont des séries régulières et achève la deuxième partie de la preuve.

La troisième partie de la preuve se réduit à comparer (+) et (++). En effet, on en déduit immédiatement

$$W^k(\hat{x}) - w^k(\hat{x}) \in \hat{m}^{\mu+1}$$

pour tout $\mu \geq 0$, ce qui entraîne $w^k(x) = W^k(x)$, comme souhaité. □

5 Application

Le but ici est de prouver la sommabilité des solutions formelles des systèmes de la forme

$$\theta_i(u) + \Lambda_{,i}u = f^i, \quad i = 1, \ldots, n, \tag{L}$$

lorsque $\Lambda(x) = 1/z(x) = \lambda(x)/x^P$, $P = (p_1, \ldots, p_n) \in \mathbb{N}^n$, $\lambda(x) \in \mathbb{C}\{x\}$ et $\lambda(0) \neq 0$. Bien évidemment, l'hypothèse essentielle est que le système a des données convergentes et est intégrable. Autrement dit, les $f^i(x) \in \mathbb{C}\{x\}$ sont des séries convergentes satisfaisant les conditions d'intégrabilité

$$(\theta_i + \Lambda_{,i})(f^j) = (\theta_j + \Lambda_{,j})(f^i), \quad i, j = 1, \ldots, n. \tag{LCI}$$

Une approche directe en choisissant une coordonnée x^i et en travaillant comme dans le cas d'une variable complexe est certainement possible, mais il est alors difficile de mettre en évidence les propriétés de sommabilité. Nous allons plutôt faire appel à la notion de série formelle régulière utilisant le théorème A comme outil fondamental.

Dans les paragraphes suivants, nous n'aurons la possibilité que d'esquisser l'interprétation en termes de sommabilité à plusieurs variables complexes, dont la théorie n'en est qu'a ses débuts (voir [10]). Essentiellement, l'idée est que le système (L) admet une unique solution formelle régulière par rapport à $z(x)$ de la forme

$$u(\hat{x}) = \sum_{\mu=0}^{+\infty} u_\mu(x) z(x)^\mu$$

mais la liberté dans le choix des $u_\mu(x)$ permet de produire une représentation ayant $u_{\mu+1}(x) = c\mu!$ pour une constante $c \in \mathbb{C}$. Nous avons donc

$$u(\hat{x}) = u_0(x) + c \sum_{\mu=0}^{+\infty} \mu! z(x)^\mu.$$

Or, une telle représentation est clairement liée à la solution formelle de l'équation d'Euler en une variable complexe, $\sum_{\mu=0}^{+\infty} \mu! \hat{\zeta}^\mu$, dont le comportement en termes de sommabilité est bien connu. Au moyen de constructions (que nous n'aurons pas la place d'expliquer en détail, renvoyant le lecteur à [10]) faisant intervenir les coordonnées $(x, \hat{\zeta})$, il devient possible de reproduire le procédé de resommation par accélération Borel/Laplace bien connu en une variable complexe (voir [9], [3], [5], [7]).

Rappelons les notations $x = (x^1, \ldots, x^n), \theta_i = x^i \partial/\partial x^i, \varphi_{,i} = \theta_i(\varphi) = x^i \partial\varphi/\partial x^i$ qui seront utilisées systématiquement.

5.1 Propriétés structurelles

Remarquons d'abord que

$$z_{,i} = \left(p_i - \frac{\lambda_{,i}}{\lambda}\right) z,$$

ce qui prouve que $z_{,i}/z \in \mathbb{C}\{x\}$. Puisque $\lambda_{,i} = \theta_i(\lambda) = x^i \partial\lambda/\partial x^i$, $z_{,i}/z$ est une série convergente inversible dès lors que $p_i \neq 0$.

Dans l'application que nous considérons ici, nous admettons $P = (1, \ldots, 1)$, i.e. $x^P = x_1 \ldots x_n$. En particulier chaque $z_{,i}/z$ est inversible, ce qui nous permet de transformer le système (L) de la manière suivante.

En introduisant les champs logarithmiques $X_j = (z/z_{,j})\theta_j$ et les séries convergentes $g^j = (z/z_{,j}) f^j$, le système (L) devient le système formé par les équations

$$zX_i(u) - u = zg^i \qquad (L_i)$$

pour $i = 1, \ldots, n$. Nous remarquons que ces équations sont de la forme (S_X) considérée dans le théorème A, avec $d = 1$ et des fonctions $\Phi_i(x, u) = u + z(x) g^i(x)$ affines. Il est donc clair que $\Phi_i(0, 0) = 0$ et $(\partial\Phi_i/\partial u)(0, 0) = 1$.

Puisque chaque équation L_i satisfait les hypothèses du théorème A, nous disposons alors d'une et une seule solution formelle régulière relativement à $z(x)$, i.e.

$$u_i(\hat{x}) = \sum_{q=0}^{+\infty} u_{iq}(x) z(x)^q, \quad u_{iq}(x) \in \mathbb{C}\{x\}.$$

Puisque $X_i z = z(X_i + 1)$, il est clair que

$$zX_i(u_i) - u_i = -u_{i0} + \sum_{q=0}^{+\infty} z^{q+1}[(X_i + q)(u_{iq}) - u_{i(q+1)}]$$

et nous devons donc avoir

$$\sum_{q=0}^{+\infty} z^{q+1}[(X_i + q)(u_{iq}) - u_{i(q+1)}] - zg^i = u_{i0}.$$

Or, dû au choix de représentation comme série régulière, nous pouvons imposer des conditions additionnelles aux u_{qi}, ce que nous ferons de manière à simplifier le problème. Supposons donc que $u_{i(q+1)}(x) = b_{iq} \in \mathbb{C}$ sont des constantes pour $q \geq 0$. Cela entraîne

$$u_{i0} = z[X_i(u_{i0}) - b_{i0}] + \sum_{q=0}^{+\infty} z^{q+2}[(q+1)b_{iq} - b_{i(q+1)}] - zg^i.$$

Nous pouvons donc admettre $b_{iq} = q!c_i$, ce qui élimine la somme infinie et donne

$$zX_i(u_{i0}) - u_{i0} = (g^i + c_i)z.$$

Les conditions imposées aux u_{iq} sont donc admissibles si nous pouvons prouver que cette équation admet une solution convergente lorsque $c_i \in \mathbb{C}$ est convenablement choisie. Or, il est clair que si

$$zX_i(u_{i0}) - u_{i0} = zg_0^i \tag{$*$}$$

avec $u_{i0}, g_0^i \in \mathbb{C}\{x\}$, alors nous avons une condition nécessaire

$$0 = \frac{1}{2i\pi} \int_{|t|=r} e^{1/z('x,t)} g_0^i('x,t) \frac{z_{,i}('x,t)}{z('x,t)} \frac{dt}{t} \tag{$**$}$$

où $x = ('x, x^i)$ désigne un regroupement des coordonnées et $r > 0$ est convenablement petit.

Il est remarquable que cette condition soit également suffisante (voir [10, Thm. 3.25]). En effet, il est possible de choisir un petit polydisque compact Δ et des $0 < \kappa < 1$, $r > 0$ tels que $|z(x)| \leq \kappa |z('x,t)|$ lorsque $x \in \Delta$ et $|t| = r$. Dans ces conditions, si $(**)$ est satisfaite, alors la formule intégrale

$$u_{i0}(x) = \frac{z(x)}{2i\pi} \int_{|t|=r} \int_0^1 e^{(1-h)/z('x,t)} \frac{g_0^i('x,t)}{z('x,t) - hz(x)} \frac{z_{,i}('x,t)}{z('x,t)} \frac{dt}{t} dh$$

se prolonge analytiquement et définit une série convergente $u_{i0}(x) \in \mathbb{C}\{x\}$ qui est l'unique solution de $(**)$ satisfaisant $u_{i0}(0) = 0$.

Il ne reste donc qu'a vérifier que l'on peut choisir $c_i \in \mathbb{C}$ de manière à ce que $(**)$ soit vérifiée. Or, posons

$$c_i('x) = -\frac{1}{2i\pi} \int_{|t|=r} e^{1/z('x,t)} g^i('x,t) \frac{z_{,i}('x,t)}{z('x,t)} \frac{dt}{t}.$$

Utilisant les conditions d'intégrabilité (LCI), on constate que $\theta_j(c_i) = 0$ et on conclut que $c_i('x) = c_i \in \mathbb{C}$ est en réalité une constante. Or, il n'est pas difficile de vérifier que

$$\frac{1}{2i\pi} \int_{|t|=r} e^{1/z('x,t)} \frac{z_{,i}('x,t)}{z('x,t)} \frac{dt}{t} = p_i = 1$$

et donc la constante c_i ainsi choisie permet de définir $g_0^i = g^i + c_i \in \mathbb{C}\{x\}$ satisfaisant $(**)$.

Regroupant ces conclusions, nous avons vu que le théorème A nous assure l'existence d'une solution régulière $u_i(\hat{x})$ de (L_i), i.e. admettant la représentation

$$u_i(\hat{x}) = \sum_{q=0}^{+\infty} u_{iq}(x) z(x)^q, \quad u_{iq}(x) \in \mathbb{C}\{x\},$$

que nous pouvons alors choisir être de la forme

$$u_{i0}(x) = \frac{z(x)}{2i\pi} \int_{|t|=r} \int_0^1 e^{(1-h)/z('x,t)} \frac{g_0^i('x,t)}{z('x,t) - hz(x)} \frac{z_{,i}('x,t)}{z('x,t)} \frac{dt}{t} dh$$
$$u_{i(q+1)}(x) = c_i q! \in \mathbb{C}, \quad c_i = -\frac{1}{2i\pi} \int_{|t|=r} e^{1/z('x,t)} g^i('x,t) \frac{z_{,i}('x,t)}{z('x,t)} \frac{dt}{t}.$$

Nous pouvons maintenant emprunter les arguments de [4, Lemma 3.1.] en les adaptant à la situation du théorème A. Introduisons donc les séries formelles régulières

$$w_{ij} = zX_i(u_j) - u_j - zg^i.$$

Alors, utilisant (LCI) on déduit

$$\left(zX_j - 1 - \frac{\Lambda_{,ij}}{\Lambda_{,j}\Lambda_{,i}} \right)(w_{ij}) = \left(zX_i - \Lambda_{,i} - \frac{\Lambda_{,ij}}{\Lambda_{,i}\Lambda_{,j}} \right)(w_{jj}).$$

Un calcul rapide montre que $\Lambda_{,ij}/(\Lambda_{,j}\Lambda_{,i}) \in z\mathbb{C}\{x\}$. Or, par hypothèse $w_{jj} = 0$, donc la série régulière w_{ij} est solution d'une équation linéaire homogène dans les conditions du théorème A. Par unicité, on conclut que $w_{ij} = 0$. Autrement dit, chaque w_j satisfait également les équations (L_i), même si $i \neq j$. De nouveau par la partie unicité du théorème A, on conclut que $u_i = u_j$ pour tous $i, j = 1, \ldots, n$.

5.2 Propriétés de sommabilité

Nous disposons donc d'une et une seule solution régulière $u(\hat{x})$ de (L) satisfaisant $u(0) = 0$. De plus, utilisant la liberté dans le choix des représentations régulières, nous avons produit, pour chaque $i = 1, \ldots, n$ des représentations de la forme

$$u(\hat{x}) = u_{i0}(x) + c_i \sum_{q=0}^{+\infty} q!z(x)^{q+1}, \quad c_i \in \mathbb{C}, u_{i0}(x) \in \mathbb{C}\{x\}.$$

Nous remarquons maintenant que les identités

$$u_{i0}(x) + c_i \sum_{q=0}^{+\infty} q!z(x)^{q+1} = u_{j0}(x) + c_j \sum_{q=0}^{+\infty} q!z(x)^{q+1}$$

ne sont possibles que si $c_i = c_j$, $u_{i0}(x) = u_{j0}(x)$, ce qui élimine l'ambiguïté dans le choix de $i = 1, \ldots, n$. Nous pouvons donc écrire

$$u(\hat{x}) = U(x, z(x)), \quad U(x, \hat{\zeta}) = \sum_{q=0}^{+\infty} u_\mu(x)\hat{\zeta}^\mu,$$

avec $u_0(x) \in \mathbb{C}\{x\}$ et $u_{\mu+1}(x) = c\mu!$.

Il est alors clair que la transformation formelle de Borel généralisée à plusieurs variables (cf. remarques dans [10]) vérifie

$$\begin{aligned}
\varphi(x, \hat{s}) &= \frac{1}{2i\pi} \int_{\hat{\zeta}\equiv 0} e^{\hat{s}/\hat{\zeta}} U(x, \hat{\zeta}) \frac{d\hat{\zeta}}{\hat{\zeta}^2} \\
&= u_0(x)\delta(\hat{s}) \oplus \sum_{\mu=0}^{+\infty} \frac{u_{\mu+1}(x)}{\mu!} \hat{s}^\mu \\
&= u_0(x)\delta(\hat{s}) \oplus c \sum_{\mu=0}^{+\infty} \hat{s}^\mu \\
&= u_0(x)\delta(\hat{s}) \oplus \frac{c}{1-\hat{s}}
\end{aligned}$$

Nous avons donc un prolongement analytique à $s \in \mathbb{C} \setminus \{0, 1\}$ et la formule de resommation par transformation de Laplace généralisée (cf. remarques dans [10]) suivante

$$\begin{aligned}
u(x, \xi) &= \frac{\xi}{z(x)} \int_0^{+\infty} e^{-h\xi/z(x)} \varphi(x, h\xi) dh \\
&= u_0(x) + c \frac{\xi}{z(x)} \int_0^{+\infty} \frac{e^{-h\xi/z(x)}}{1 - \xi h} dh
\end{aligned}$$

qui définit une fonction holomorphe dans un domaine de la forme

$$\max_j |x^j| < r, \quad \xi \in \mathbb{C} \setminus [0, +\infty[, \ \Re\left(\frac{\xi}{z(x)}\right) > 0$$

et qui, de plus, est une fonction localement constante de ξ. En choisissant des valeurs particulières ξ_α, on obtient des sommes $u_\alpha(x)$ définies et holomorphes sur des domaines ouverts Ω_α dans $\mathbb{C}^n$ qui sont la généralisation au cas de plusieurs variables des grands secteurs de la théorie de la sommabilité en une variable complexe (voir [10]).

La fonction $u(x, \xi)$ ainsi définie (où si l'on préfère la cochaîne des u_α) est la somme (au sens de [10]) de la solution formelle régulière $u(\hat{x})$. En particulier son développement asymptotique Gevrey généralisé (au sens de [10]) est précisément $u(\hat{x})$. Par conséquent, on peut vérifier soit par le lemme de Watson à plusieurs variables complexes [10][Thm. 2.61], soit par inspection directe que $u(x, \xi)$ est l'unique solution holomorphe de (L) que l'on obtient par resommation de $u(\hat{x})$.

Remarquons par ailleurs que l'automorphisme de Stokes correspond à l'obstruction présentée par $\xi \notin [0, +\infty[$ et peut être interprété comme l'hyperfonction réelle (à paramètres) définie sur $\xi \in [0, +\infty[$ par

$$u(x, \xi + i0) - u(x, \xi - i0) \equiv v(x, \xi + i0)$$

où $v(x, \xi + i0) = e^{-1/z(x)}$ est constante (en ξ).

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Remarques algorithmiques liées au rang d'un opérateur différentiel linéaire

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Dédié à Jean Thomann
à l'occasion de son 60^e *anniversaire*

Résumé. Dans cet article nous décrivons et nous comparons différentes méthodes algorithmiques pour résoudre le problème suivant qui apparaît naturellement lors de l'implantation en calcul formel des solutions formelles d'équations différentielles et de leurs (multi-)sommes : étant donnés une série $\hat{f}(x) = \sum_{m\geq 0} u_m x^m$ solution d'une équation différentielle linéaire ordinaire et un nombre entier $r > 0$ construire une équation différentielle vérifiée par les sous-séries $\hat{f}^j(x) = \sum_{m\geq 0} u_{j+mr} x^{j+mr}$ de ses termes de r en r. Les méthodes proposées sont implantées en MAPLE et nous donnons l'adresse des programmes correspondants.

En appendice nous indiquons une démarche constructive réaliste pour le calcul effectif des invariants formels d'une équation ou d'un système.

Mots clés. Équations différentielles linéaires ordinaires, équations aux différences linéaires, extraction de sous-séries, transformation de Mellin, méthodes algorithmiques.

Abstract. This paper aims at introducing and comparing various methods for solving the following problem which arises naturally in computer algebra for differential equations : given a

series $\hat{f}(x) = \sum_{m\geq 0} u_m x^m$ solution of an ordinary linear differential equation and given an integer $r \geq 2$ find a differential equation satisfied by the sub-series $\hat{f}^j(x) = \sum_{m\geq 0} u_{j+mr} x^{j+mr}$. These methods have been implemented in MAPLE. References are given to the corresponding programs.
Possible realistic algorithms to effectively calculate the formal invariants of an ordinary linear differential system of order one and arbitrary rank are sketched in an Appendix.

Keywords. Ordinary linear differential equations, linear difference equations, extraction of sub-series, Mellin transform, algorithmic methods.

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1 Introduction

Étant donné un système différentiel linéaire

$$\frac{dY}{dx} = \frac{1}{x^{r+1}} A(x) Y$$

de dimension n à coefficients méromorphes d'ordre $r + 1$ à l'origine ($r \in \mathbb{Z}$, $A \in \mathrm{gl}(n, K)$ où $K = \mathbb{C}\{x\}$ est l'anneau des séries convergentes à l'origine et $A(0) \neq 0$) il est d'usage d'appeler *rang de Poincaré* du système le nombre r. Ce nombre n'est pas invariant par transformation de jauge méromorphe $Y = T(x)Z$ où $T \in K[1/x]$ mais il admet une valeur minimale appelée *vrai rang de Poincaré*. Les systèmes de rang de Poincaré $r \leq -1$ n'ont pas de singularité à l'origine, ceux de rang $r = 0$ ont une singularité régulière et, lorsque le vrai rang de Poincaré est strictement positif, l'origine est un point singulier irrégulier.

On dit, de même, qu'une équation différentielle

$$a_n(x) y^{(n)} + a_{n-1}(x) y^{(n-1)} + \cdots + a_0(x) y = 0$$

à coefficients méromorphes en $x = 0$ est de rang r si les coefficients $\frac{a_{n-1}}{a_n}, \ldots, \frac{a_0}{a_n}$ de l'équation normalisée ont un pôle d'ordre $r + 1$ en 0.

Lorsque le système est de vrai rang de Poincaré égal à r et qu'il porte le seul niveau r (les exponentielles d'une solution fondamentale formelle sont toutes d'ordre r), les séries solutions formelles divergentes sont r-sommables et, en particulier, de type Gevrey d'ordre r. Lorsqu'il porte plusieurs niveaux $r_1 < r_2 < \cdots < r_s \leq r$ ses solutions formelles divergentes sont au pire multisommables des niveaux $(r_1, r_2, \ldots, r_s)$ et de type Gevrey d'ordre r_1 ([M95]).

C'est au début des années 80 que, grâce à de nouvelles avancées de la théorie des équations différentielles ordinaires, s'est exprimé un intérêt croissant pour les méthodes effectives et le développement d'algorithmes liés aux singularités d'équations différentielles : calcul de solutions formelles, d'invariants méromorphes, de sommes des solutions séries divergentes,. . .

Ces dernières années, Jean Thomann en particulier s'est beaucoup investi dans le développement et l'implantation des algorithmes de sommation. Les différentes méthodes théoriques connues reposent sur des transformations intégrales de type opérateurs de Borel et de Laplace associés aux différents niveaux de l'équation. Or, dans cette famille de transformations, les opérateurs de Borel et de Laplace classiques, i.e., de niveau 1, jouent un rôle particulier qui les rend incontournables pour mener à bien les différentes étapes de calcul. Pour s'en convaincre mentionnons simplement le fait suivant : la transformée de Borel d'un opérateur différentiel à coefficients polynomiaux est un opérateur différentiel à coefficients polynomiaux auquel on peut appliquer toute la théorie des équations différentielles et, en particulier, les algorithmes numériques ; au contraire, la transformée de Borel d'ordre $r > 1$ d'un tel opérateur est une équation de r-convolution mal connue. Mais, dans le cas d'un niveau unique r, cas de la r-sommabilité, il se trouve que l'on peut se limiter à l'implantation du cas $r = 1$, cas classique de la sommabilité au sens de Borel-Laplace ; en effet, si une série $\hat{f}(x) = \sum_{m \geq 0} u_m x^m$ est r-sommable alors les sous-séries extraites $\hat{f}^j(x) = x^j \sum_{m \geq 0} u_{mr} x^{mr}$ des termes de r en r sont 1-sommables en la variable $t = x^r$ et il y a compatibilité des procédés de sommation. Dans le cas de plusieurs niveaux, J. Thomann a examiné à la fois la méthode des accélératrices (due à J. Écalle) et celle de Laplace-itérée (due à W. Balser). Ces méthodes consistent à appliquer successivement des transformations de type transformations de Borel ou de Laplace. Là encore, la tactique la plus efficace à ce jour consiste à se ramener systématiquement au niveau 1, i.e., au niveau 1 à chaque étape. Pour plus de détails on renvoie à [JNT96] et à [NT96]. On est ainsi conduit à se poser la question suivante :

Comment obtenir le plus simplement possible des équations différentielles auxquelles satisfont les $\hat{f}^j(x)$ à partir d'une équation différentielle à laquelle satisfait $\hat{f}(x)$?

Nous explicitons au paragraphe 2 un certain nombre de méthodes conduisant à une équation différentielle qui admet simultanément toutes les $\hat{f}_j$ pour solutions. Puis nous indiquons au paragraphe 4.3 comment obtenir des équations différentielles (chacune d'ordre inférieur mais plusieurs équations) auxquelles satisfont séparément chacune de séries $\hat{f}_j$.

La même question apparaît de façon sous-jacente dans les algorithmes de calcul direct, c'est-à-dire, sans sommation, des invariants méromorphes comme, par exemple, dans [L-R90] en dimension 2. Ce sont là les relations de récurrence sur les coefficients des séries $\hat{f}^j(x)$ qui sont directement utiles et on est amené à considérer le problème dual :

Comment obtenir le plus simplement possible des équations aux différences auxquelles satisfont les suites de coefficients des séries $\hat{f}^j(x)$ à partir d'une équation aux différences à laquelle satisfait la suite de coefficients de la série $\hat{f}(x)$?

Ces deux problèmes se correspondent par transformation de Mellin. Nous avons cherché, dans la mesure du possible, à présenter en parallèle des méthodes traitant directement l'équation différentielle et les méthodes analogues duales pour l'équation aux différences qui s'en déduit par transformation de Mellin. Nous montrons que les approches directes conduisent toutes à la même équation différentielle $Ly = 0$ admettant les $\hat{f}^j$ pour solutions. Les approches duales conduisent à une même équation aux différences $\Lambda\widetilde{y} = 0$. Il eût été raisonnable d'espérer que L et Λ se correspondent par transformation de Mellin. Il n'en est rien comme le confirme l'exemple de l'équation de Ramis–Sibuya que nous explicitons. Nous expliquerons pourquoi il en est ainsi au paragraphe 4.1.

Notons enfin que le développement d'algorithmes adaptés aux systèmes incite à se poser les mêmes questions en termes de systèmes. Les performances de l'algorithme 5*(1) ou même 5(1) sur l'exemple de l'équation de Ramis–Sibuya montrent d'ailleurs qu'on peut avoir intérêt à travailler avec un système plutôt qu'une équation même si la donnée initiale est une équation (*cf.* paragraphe 4.2).

Il existe des solutions connues voire programmées à ces problèmes, en particulier, sous leur forme duale ([SZ94], [N95]). À notre connaissance toutes les méthodes proposées en termes de systèmes sont originales.

Les méthodes développées ou au moins certains de leurs aspects sont valables de façon générale pour des équations ou des systèmes à coefficients méromorphes. Pour qu'elles deviennent totalement algorithmiques, nous nous restreindrons néammoins à des équations ou à des systèmes à coefficients polynomiaux ou rationnels.

La distinction que nous faisons entre des méthodes dites de saturation galoisienne et des méthodes dites d'élimination est à frontière floue dans la mesure où ces éliminations ont pour objet de réaliser la saturation galoisienne.

Les arguments galoisiens ont été à la base des travaux initiateurs de Turrittin sur la question plus générale de la réduction du rang des systèmes ([T63], [L-R01]). L'argument de Turrittin est simple et analogue à celui évoqué plus haut : si un système est de rang r en la variable x il est de rang 1 en la «variable» $t = x^r$. La simple substitution de t à x^r fait apparaître des puissances fractionnaires de t. Pour rester dans le cadre des systèmes à coefficients méromorphes il s'agit alors de saturer l'effet de ce changement de variable par l'action du groupe de Galois $\mathfrak{S}_r$ de l'équation $x^r = t$; ce faisant, on multiplie la dimension par r (du moins génériquement).

Les arguments d'élimination s'appuient sur des algorithmes de Gauss ou d'Euclide dans des algèbres de polynômes non commutatives.

L'exécution de ces algorithmes en MAPLE sur l'exemple de l'équation de Ramis–Sibuya nous amène à faire les commentaires suivants. Les temps de calcul requis sont raisonnablement comparables lorsque $r = 2$. En revanche, lorsque $r > 2$, ils deviennent significativement différents. Les algorithmes les plus coûteux sont ceux qui font un usage effectif de nombres algébriques sur le corps des coefficients (essentiellement de $\omega = e^{2\pi i/r}$) et ceux qui utilisent de l'arithmétique non commutative. Les plus performants sont ceux qui s'appuient sur la théorie de la réduction du rang des systèmes ou le produit de Hadamard en restant dans le cadre de l'élimination linéaire.

Le calcul de la complexité de ces algorithmes reste à faire. Les procédures que nous avons écrites n'ont pas, pour l'instant, été optimisées. Les observations faites sur l'exemple de l'équation de Ramis–Sibuya nous incitent d'ailleurs à ne pas porter un même intérêt à tous ces algorithmes.

Pour répondre à une demande explicite de certains utilisateurs, nous complétons cet article par un appendice sur le calcul des invariants formels des opérateurs différentiels (équations ou systèmes) et, en particulier, de leurs niveaux qui sont des valeurs privilégiées pour la réduction du rang. Il ne s'agit pas ici de présenter des méthodes nouvelles mais de rassembler et d'ordonner des résultats algorithmiques existants en un mode opératoire précis dont chacun pourra utiliser tout ou partie. Les programmes correspondants sont implantés en MAPLE et disponibles à l'adresse électronique citée en référence [BCL].

2 Approche directe dans l'anneau des opérateurs différentiels

Dans ce paragraphe nous considérons une équation différentielle linéaire

$$Dy \equiv a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = 0 \tag{1}$$

dont les coefficients $a_0, a_1, \ldots, a_n$ sont polynomiaux. Nous supposons que cette équation admet une solution série formelle $\hat{f}(x) = \sum_{m\geq 0} u_m x^m$ et nous nous proposons de déterminer une équation différentielle qui admet pour solutions les séries

$$\hat{f}^j(x) = \sum_{m\geq 0} u_{j+mr} x^{j+mr}.$$

Nous notons $\partial = \dfrac{d}{dx}$ l'opérateur de dérivation usuelle et $\mathbb{W}_1 = \mathbb{C}[x][\partial]$ l'algèbre de Weyl des opérateurs différentiels à coefficients polynomiaux à une indéterminée x dans laquelle les relations de commutation sont engendrées par l'unique relation

$$x\partial - \partial x = 1.$$

L'algèbre de Weyl $\mathbb{W}_1$ se plonge de façon canonique dans l'algèbre $\mathbb{C}(x)[\partial]$ des opérateurs différentiels à coefficients rationnels munie de la même relation de commutation. Celle-ci a l'avantage d'être euclidienne à gauche (et à droite), donc aussi principale à gauche (et à droite) ([Ore33]).

Quitte à multiplier D par une puissance convenable de x, nous supposons désormais que le 0-poids de D (*cf.* 5.1) est positif ou nul et que, par conséquent, D est polynomial en x et en l'opérateur d'Euler $\delta = x\partial$. Les opérateurs D considérés appartiennent alors à l'algèbre $\mathbb{A}_1 = \mathbb{C}[x][\delta]$ si on utilise la dérivation δ ou à l'image $\mathbb{A}_1'$ de celle-ci dans $\mathbb{W}_1$ si on utilise la dérivation ∂. Les relations de commutation dans $\mathbb{A}_1$ sont engendrées par l'unique relation

$$x\delta - \delta x = x.$$

Ceux des algorithmes ci-dessous qui sont écrits avec ∂ restent valables, à des modifications mineures de notation près, avec δ.

2.1 Calcul dans le cadre des équations

2.1.1 Par saturation galoisienne. On fait agir le groupe de Galois $\mathfrak{S}_r$ de l'équation $x^r = t$ directement sur l'équation différentielle $Dy = 0$. En itérant l'action du générateur $\breve{\omega} : x \mapsto \omega x$ (choisissons $\omega = e^{2\pi i/r}$ par exemple) on obtient des équations

$$\begin{cases} D_0 y = 0 \text{ où } D_0 = D, \text{ qui admet la solution } \hat{f}(x), \\ D_1 y = 0 \text{ où } D_1 = \breve{\omega}.D, \text{ qui admet la solution } \hat{f}(\omega x), \\ \qquad \vdots \\ D_{r-1} y = 0 \text{ où } D_{r-1} = \breve{\omega}^{r-1}.D, \text{ qui admet la solution } \hat{f}(\omega^{r-1} x). \end{cases}$$

On peut voir $D_0, \ldots, D_{r-1}$ comme des éléments de $C(x)[\partial]$ et considérer, dans cet anneau principal à gauche, un plus petit commun multiple à gauche L (en abrégé ppcmg). L'équation différentielle $Ly = 0$ admet les solutions $\hat{f}(x), \hat{f}(\omega x), \ldots, \hat{f}(\omega^{r-1} x)$, donc les solutions souhaitées

$$\begin{cases} \hat{f}^0(x) = \dfrac{1}{r}\Big(\hat{f}(x) + \hat{f}(\omega x) + \cdots + \hat{f}(\omega^{r-1} x)\Big), \\ \hat{f}^1(x) = \dfrac{1}{r}\Big(\hat{f}(x) + \dfrac{1}{\omega}\hat{f}(\omega x) + \cdots + \dfrac{1}{\omega^{r-1}}\hat{f}(\omega^{r-1} x)\Big), \\ \qquad \vdots \\ \hat{f}^{r-1}(x) = \dfrac{1}{r}\Big(\hat{f}(x) + \dfrac{1}{\omega^{r-1}}\hat{f}(\omega x) + \cdots + \dfrac{1}{\omega^{(r-1)(r-1)}}\hat{f}(\omega^{r-1} x)\Big). \end{cases}$$

Pour être précis, la notation L désigne désormais l'unique ppcmg appartenant à $\mathbb{A}'_1$ (on «chasse» les dénominateurs) qui soit unitaire (le coefficient de la plus haute puissance de ∂ est un polynôme unitaire) et primitif (les différents coefficients des puissances de ∂ sont premiers entre eux). Par construction, l'équation $Ly = 0$ est l'équation d'ordre minimal qui admet pour solutions toutes celles de l'équation initiale $Dy = 0$ et qui est stable sous l'action de $\mathfrak{S}_r$. Il n'y aucune raison pour qu'elle soit l'équation d'ordre minimal qui admette les $\hat{f}_j$ pour solutions, pas plus d'ailleurs que $Dy = 0$ n'est supposée être l'équation d'ordre minimal qui admet $\hat{f}$ pour solution. Voir, par exemple, aux paragraphes 4.2 et 4.3, le cas de l'équation de Ramis–Sibuya qui y est traité en détail.

On est ainsi conduit à l'algorithme suivant :

Algorithme : version 1.

1. Pour $j = 0, 1, \ldots, r-1$, déterminer l'opérateur D_j en substituant $\omega^j x$ à x et $\omega^{-j}\partial$ à ∂ (ou δ à lui-même) dans l'opérateur D.

2. Dans l'algèbre $\mathbb{C}(x)[\partial]$ (ou dans $\mathbb{C}(x)[\delta]$) déterminer un plus petit commun multiple à gauche de $D_0, D_1, \ldots, D_{r-1}$.
3. Si on le souhaite, déterminer l'unique représentant unitaire et primitif de ce ppcmg dans $\mathbb{A}'_1$, d'où L.

La procédure **LCLM** du paquetage DEtools de MAPLE fournit un ppcmg d'une liste d'opérateurs ; le résultat est donné avec un coefficient de tête égal à 1. On obtient un représentant de ce ppmcg dans l'anneau $\mathbb{A}'_1$ en chassant les dénominateurs.

La procédure **skew_gcdex** du paquetage Ore_algebra de MAPLE fournit un pgcdd de deux polynômes p et q et la relation de Bézout qu'ils vérifient avec des coefficients dans $\mathbb{A}'_1$, i.e., sans dénominateur. Elle permet aussi de calculer leur ppcmg bien qu'elle ne le donne pas directement. La commande **skew_gcdex**(p, q, Dx, A) où $A :=$ **skew_algebra**(diff $= [Dx, x]$) restitue $[g, a, b, u, v]$ où $g = ap + bq$ est le pgcdd de p et de q et où $up = -vq$ est le ppcmg de p et de q. On obtient le ppcmg cherché par la commande **skew_product**(u, p, A).

2.1.2 Par élimination. On peut conduire l'élimination de diverses manières. Nous en présentons deux variantes.

• Première variante :

On introduit une nouvelle indéterminée Ω. Si l'on remplace x par Ωx et ∂ par $\Omega^{-1}\partial$ (ou δ par lui-même), quitte à multiplier au préalable D par une puissance convenable de x comme ci-dessus, on obtient un polynôme $P(\Omega)$ à coefficients dans $\mathbb{A}'_1$. L'existence d'un ppcmg de deux éléments quelconques de l'anneau $\mathbb{C}(x)[\partial]$ permet de définir un corps des fractions à gauche non commutatif F_g formé des quotients formels $A(x,\partial)^{-1}B(x,\partial)$ pour $A(x,\partial) \neq 0$ et $B(x,\partial) \in \mathbb{C}(x)[\partial]$ muni de la relation d'équivalence évidente ([Ore33] I.3, [Co] 0.5). On peut ainsi plonger de façon canonique $\mathbb{A}'_1$ et $\mathbb{C}(x)[\partial]$ dans le corps F_g. Dans l'anneau euclidien $F_g[\Omega]$ (ici, c'est F_g qui est non-commutatif alors que Ω commute avec les éléments de F_g) les polynômes $P(\Omega)$ et $\Omega^r - 1$ sont premiers entre eux puisque, pour $j = 0, 1, \ldots, r-1$, $P(\omega^j) = D_j \neq 0$. Ils vérifient donc une relation de Bézout

$$S(\Omega)P(\Omega) + T(\Omega)(\Omega^r - 1) = G$$

où, quitte à chasser les dénominateurs, on peut supposer que G est dans $\mathbb{A}'_1$ alors que S et T sont des polynômes de $\mathbb{A}'_1[\Omega]$ de degré respectivement inférieur à r et au degré de P.

Notons que, quitte à chasser les dénominateurs à chaque étape, l'algorithme d'Euclide peut être conduit dans l'algèbre $\mathbb{A}'_1[\Omega]$ en effectuant des pseudo-divisions.

En spécifiant Ω en les diverses racines $r^{\grave{e}mes}$ de l'unité ω^j on en déduit que G est un multiple à gauche ML du ppcmg L obtenu précédemment. Par ailleurs, $S(\Omega)$ est de degré $\leq r-1$ en Ω ; le même facteur M divise à gauche dans $\mathbb{C}(x)[\partial]$ tous les $S(\omega^j)$ donc aussi toutes leurs combinaisons linéaires et, en particulier, les coefficients de $S(\Omega)$. Il en résulte que M divise à gauche $T(\Omega)(\Omega^r - 1)$, donc également le coefficient constant $T(0)$ de $T(\Omega)$ puis, par récurrence, tous les coefficients de T. Après simplification par M, factorisation du contenu commun des coefficients et normalisation, on obtient donc la relation $G = uL$ où $u \in \mathbb{C}[x]$.

Algorithme : version 2.

1. Déterminer le polynôme $P \in \mathbb{A}'_1[\Omega]$ en substituant Ωx à x et $\Omega^{-1}\partial$ à ∂ (ou δ à lui-même) dans l'opérateur D.
2. Grâce à un algorithme d'Euclide dans l'algèbre $F_g[\Omega]$ où F_g désigne le corps des fractions à gauche de $\mathbb{C}(x)[\partial]$ déterminer la relation de Bézout

$$S(\Omega)P(\Omega) + T(\Omega)(\Omega^r - 1) = uL$$

dans laquelle S et T sont dans $\mathbb{A}'_1[\Omega]$, leurs coefficients sans contenu commun dans $\mathbb{A}'_1$, $u \in \mathbb{C}[x]$ et $L \in \mathbb{A}'_1$ est primitif et normalisé.

• Deuxième variante :

Exprimons D en la dérivation δ :

$$D = b_n(x)\delta^n + b_{n-1}(x)\delta^{n-1} + \cdots + b_0(x)$$

où, conformément à l'hypothèse que le 0-poids de D est positif ou nul, les coefficients $b_j(x)$ appartiennent à $\mathbb{C}[x]$.

Il s'agit de déterminer le plus petit multiple à gauche de D dont les coefficients ne contiennent que des puissances de x^r, i.e., en posant $t = x^r$, un multiple à gauche de D d'ordre minimal et appartenant à l'algèbre $\mathbb{C}[t][\delta]$. Pour cela, on procède par élimination de Gauß. On plonge à nouveau $\mathbb{C}[x]$ dans son corps des fractions $\mathbb{C}(x)$ puis on identifie $\mathbb{C}(x)$ au $\mathbb{C}(t)$-espace vectoriel $\mathbb{C}(t)_{r-1}[x] \simeq \mathbb{C}(t)[x]/(x^r - t)$ de base $1, x, \ldots, x^{r-1}$ (voir la remarque 2.3 ci-dessous). Le quotient de $\mathbb{C}(x)[\delta]$ par l'idéal engendré par D à droite est alors un $C(t)$-espace vectoriel de dimension finie nr engendré par les monômes $x^\ell\delta^k$ pour $0 \leq \ell \leq r-1$ et $0 \leq k \leq n-1$. Notons U_j le représentant de δ^j dans cette base. Une élimination de Gauss entre les U_j successifs, et ce jusqu'à $j = nr$ au plus, conduit à une relation de dépendance $\mathbb{C}(t)$-linéaire $\sum L_j(t)U_j = 0$. L'opérateur $L = \sum L_j(x^r)\delta^j$ est un multiple de D invariant par $\mathfrak{S}_r$ et d'ordre minimal. C'est donc, après suppression des dénominateurs et normalisation, l'opérateur L.

Algorithme : version 3.

1. L'opérateur D étant supposé de 0-poids positif ou nul, l'exprimer dans l'algèbre $\mathbb{A}_1 = \mathbb{C}[x][\delta]$:

$$D = b_n(x)\delta^n + b_{n-1}(x)\delta^{n-1} + \cdots + b_0(x) \text{ et poser } t = x^r.$$

2. Remplacer D par l'opérateur unitaire $R = \delta^n + \dfrac{b_{n-1}(x)}{b_n(x)}\delta^{n-1} + \cdots + \dfrac{b_0(x)}{b_n(x)}$ de $\mathbb{C}(x)[\delta]$ et exprimer les coefficients de ce dernier dans la base $(x^\ell)_{0\leq\ell\leq r-1}$ du $\mathbb{C}(t)$-espace vectoriel $\mathbb{C}(x)$.

3. Déterminer les représentants U_j des puissances successives δ^j de δ dans la base $(x^\ell \delta^k)_{\substack{0\le\ell\le r-1\\0\le k\le n-1}}$ du $\mathbb{C}(t)$-espace vectoriel $\mathbb{C}(x)[\delta]/(D)$.

4. Par un algorithme de Gauss dans ce $\mathbb{C}(t)$-espace vectoriel et en chassant les dénominateurs déterminer la première relation $\sum L_j(t)U_j = 0$ de dépendance $\mathbb{C}[t]$-linéaire entre les U_j.

5. $L = \sum L_j(x^r)\delta^j$ à normalisation près.

Remarque 2.1. Le point 2 de l'algorithme ci-dessus s'obtient par division euclidienne dans l'anneau $\mathbb{C}(t)[x]$. Nous détaillons cet algorithme dans la remarque (2.3), variante 1, ci-dessous. Ce calcul revient, de fait, à extraire les termes de r en r dans le développement en série d'une fraction rationnelle.

2.2 Calcul par l'intermédiaire des systèmes

Dans ce paragraphe nous conservons pour donnée initiale l'équation différentielle $Dy = 0$ et nous proposons des algorithmes qui opèrent sur son système compagnon. Ces mêmes algorithmes peuvent s'appliquer directement à un système sans réduction à une forme compagnon si telle était la donnée initiale.

2.2.1 Par saturation galoisienne directe. Considérons un système méromorphe quelconque (pas nécessairement sous forme compagnon) à l'origine

$$\Delta Y \equiv \frac{dY}{dx} - \frac{1}{x^{r+1}}A(x)Y = 0.$$

(A étant quelconque, r n'est pas en général le rang de Poincaré du système mais cette notation est plus simple pour la suite). L'action itérée de $\breve{\omega} : x \mapsto \omega x$ donne naissance à r systèmes $\Delta_j Y \equiv \dfrac{dY}{dx} - \dfrac{1}{x^{r+1}}A(\omega^j x)Y, \quad j = 0, 1, \ldots, r-1$, où, si $Y(x)$ est une solution de $\Delta Y = 0$, on voit que $Y(\omega^j x)$ est solution de $\Delta_j Y = 0$ et le vecteur inconnu $\widetilde{Y} = \begin{bmatrix} Y(x) \\ Y(\omega x) \\ \vdots \\ Y(\omega^{r-1}x) \end{bmatrix}$ satisfait à l'équation matricielle de dimension nr

$$\widetilde{\Delta}\widetilde{Y} \equiv \frac{d\widetilde{Y}}{dx} - \frac{1}{x^{r+1}}\widetilde{A}(x)\widetilde{Y} = 0 \quad \text{où} \quad \widetilde{A}(x) = \bigoplus_{j=0}^{r-1} A(\omega^j x).$$

Supposons désormais que $\Delta Y = 0$ soit le système compagnon de l'équation (1) : $Dy = 0$.

Alors, à toute solution $\hat{f}(x)$ de (1) correspond la solution

$$\widehat{F}(x) = \begin{bmatrix} \hat{f}(x) \\ \hat{f}'(x) \\ \vdots \\ \hat{f}^{(r-1)}(x) \end{bmatrix}$$

de $\Delta Y = 0$ et les solutions

$$\widetilde{F}(x) = \begin{bmatrix} \lambda_0 \widehat{F}(x) \\ \lambda_1 \widehat{F}(\omega x) \\ \vdots \\ \lambda_{r-1} \widehat{F}(\omega^{r-1} x) \end{bmatrix}$$

de $\widetilde{\Delta}\widetilde{Y} = 0$ pour des constantes complexes $\lambda_0, \lambda_1, \ldots, \lambda_{r-1}$ arbitraires.

Considérons les vecteurs-lignes $E_1 = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}$ de dimension n et $\widetilde{E} = \begin{bmatrix} E_1 & E_1 & \ldots & E_1 \end{bmatrix}$ de dimension nr. La méthode du vecteur cyclique (*cf.* 5.3) appliquée au système $\widetilde{\Delta}\widetilde{Y} = 0$ à partir du vecteur $\widetilde{E}$ fournit une équation différentielle qui admet pour solution toutes les combinaisons linéaires possibles de $\hat{f}(x), \hat{f}(\omega x), \ldots, \hat{f}(\omega^{r-1}x)$. Cet espace vectoriel est aussi engendré par les séries $\hat{f}^0(x), \hat{f}^1(x), \ldots, \hat{f}^{r-1}(x)$ auxquelles nous nous intéressons. Par construction, la méthode fournit l'équation différentielle d'ordre minimal admettant pour solutions toutes celles de l'équation (1) et leurs orbites sous l'action du groupe de Galois $\mathfrak{S}_r$ de $x^r = t$. On obtient donc ici encore la même équation $Ly = 0$ que celle obtenue par la méthode précédente où L est le ppcmg des opérateurs $D_0, D_1, \ldots, D_{r-1}$.

En particulier, et comme on le verra sur l'exemple de l'équation de Ramis–Sibuya, l'équation $Ly = 0$ n'est pas toujours d'ordre maximal nr ; autrement dit, le vecteur $\widetilde{E}$ n'est pas nécessairement cyclique.

Ces observations nous conduisent à l'algorithme suivant :

Algorithme : version 4.

1. Écrire le système compagnon de l'équation (1), à savoir,

$$\Delta Y \equiv \frac{dY}{dx} - \frac{1}{x^{r+1}} A(x) Y$$

où

$$\frac{1}{x^{r+1}}A(x)=\begin{bmatrix} 0 & 1 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & 0 & 1 \\ \dfrac{-a_0}{a_n} & \cdots & & \dfrac{-a_{n-2}}{a_n} & \dfrac{-a_{n-1}}{a_n} \end{bmatrix}.$$

2. Écrire le système somme directe

$$\widetilde{\Delta}\widetilde{Y}\equiv x^{r+1}\frac{d\widetilde{Y}}{dx}-\widetilde{A}(x)\widetilde{Y}=0 \quad \text{où} \quad \widetilde{A}(x)=\bigoplus_{j=0}^{r-1}A(\omega^j x).$$

3. Appliquer l'algorithme du vecteur cyclique au système $\widetilde{\Delta}\widetilde{Y}=0$ et au vecteur $\widetilde{E}$, d'où l'obtention de l'opérateur L après normalisation éventuelle.

2.2.2 Par réduction du rang. Il s'agit, ici encore, d'une méthode par saturation galoisienne mais celle-ci est partie intégrante de la réduction du rang dont nous utilisons maintenant les propriétés.

Notons $\mathcal{Y}(x)$ une matrice fondamentale de solutions d'un système quelconque $\Delta Y=0$. Le système r-réduit associé à $\Delta Y=0$ est l'unique système de la variable $t=x^r$, méromorphe en 0 et admettant les solutions $\begin{bmatrix} \mathcal{Y}(x) \\ x^{-1}\mathcal{Y}(x) \\ \vdots \\ x^{-(r-1)}\mathcal{Y}(x) \end{bmatrix}$ pour un choix donné $x=t^{1/r}$ d'une racine $r^{\text{ème}}$ de t. Ce système admet pour matrice fondamentale de solutions la matrice

$$\mathbf{V}(t)=\begin{bmatrix} \mathcal{Y}(t^{1/r}) & \mathcal{Y}(\omega t^{1/r}) & \cdots & \mathcal{Y}(\omega^{r-1}t^{1/r}) \\ (t^{1/r})^{-1}\mathcal{Y}(t^{1/r}) & (\omega t^{1/r})^{-1}\mathcal{Y}(\omega t^{1/r}) & \cdots & (\omega^{r-1}t^{1/r})^{-1}\mathcal{Y}(\omega^{r-1}t^{1/r}) \\ \vdots & \vdots & & \vdots \\ (t^{1/r})^{-(r-1)}\mathcal{Y}(t^{1/r}) & (\omega t^{1/r})^{-(r-1)}\mathcal{Y}(\omega t^{1/r}) & \cdots & (\omega^{r-1}t^{1/r})^{-(r-1)}\mathcal{Y}(\omega^{r-1}t^{1/r}) \end{bmatrix}$$

pour un choix $x=t^{1/r}$ quelconque d'une racine $r^{ème}$ de t.

Pour $\ell=0,1,\dots,r-1$, notons $A^{\ell}(t)=\sum_m A_{\ell+rm}t^{-m}$ de sorte que

$$A(x)=A^0(x^r)+xA^1(x^r)+\cdots+x^{r-1}A^{r-1}(x^r).$$

Le système r-réduit associé à $\Delta Y = 0$ s'écrit ([L-R01], Theorem 1.1)

$$\widetilde{\Delta}^r \boldsymbol{Y} \equiv rt^2 \frac{d\boldsymbol{Y}}{dt} - \Big(\boldsymbol{A}(t) - \bigoplus_{j=0}^{r-1} jtI\Big)\boldsymbol{Y} = 0$$

où I désigne la matrice identité de dimension n et où

$$\boldsymbol{A}(t) = \begin{bmatrix} A^0(t) & tA^{r-1}(t) & tA^{r-2}(t) & \cdots & \cdots & \cdots & tA^1(t) \\ A^1(t) & A^0(t) & tA^{r-1}(t) & \ddots & & & \vdots \\ A^2(t) & A^1(t) & A^0(t) & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & A^0(t) & tA^{r-1}(t) & tA^{r-2}(t) \\ \vdots & & & \ddots & A^1(t) & A^0(t) & tA^{r-1}(t) \\ A^{r-1}(t) & \cdots & \cdots & \cdots & A^2(t) & A^1(t) & A^0(t) \end{bmatrix}$$

Supposons maintenant, à nouveau, que $\Delta Y = 0$ soit le système compagnon de l'équation (1) : $Dy = 0$ admettant une solution formelle $\hat{f}(x) = \sum_{m\geq 0} u_m x^m$. Alors, compte tenu de la forme de la solution fondamentale $\boldsymbol{V}$, le système $\widetilde{\Delta}^r \boldsymbol{Y} = 0$ admet, pour tout j, une solution dont la première composante est égale à $\sum_{m\geq 0} u_{j+mr} t^m$. Dans un souci d'homogénéité avec les sections précédentes, revenons à la variable $x = t^{1/r}$ et aux séries $\hat{f}^j(x) = \sum_{m\geq 0} u_{j+mr} x^{j+mr}$ en posant $\boldsymbol{X}(x) = \big(\bigoplus_{j=0}^{r-1} x^j I\big)\boldsymbol{Y}(x^r)$. On obtient le système

$$x^{r+1} \frac{d\boldsymbol{X}}{dx} - \boldsymbol{B}(x)\boldsymbol{X} = 0 \tag{2}$$

où la matrice $\boldsymbol{B}$ s'écrit

$$\boldsymbol{B}(x) = \begin{bmatrix} A^0(x^r) & x^{r-1}A^{r-1}(x^r) & \cdots & \cdots & x^2A^2(x^r) & xA^1(x^r) \\ xA^1(x^r) & A^0(x^r) & \ddots & & & x^2A^2(x^r) \\ x^2A^2(x^r) & xA^1(x^r) & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & A^0(x^r) & x^{r-1}A^{r-1}(x^r) \\ x^{r-1}A^{r-1}(x^r) & \cdots & \cdots & \cdots & xA^1(x^r) & A^0(x^r) \end{bmatrix}$$

Ce système admet, pour $j = 0, 1, \dots, r-1$, une solution dont la première composante est égale à $\hat{f}^j(x)$. Le résultat est d'ailleurs vrai pour toute composante d'indice $1 \bmod n$. On peut alors proposer l'algorithme suivant :

Algorithme : version 5.

1. Écrire le système compagnon de l'équation (1) à savoir

$$x^{r+1}\frac{dY}{dx} = A(x)Y \quad \text{où} \quad A(x) = x^{r+1}\begin{bmatrix} 0 & 1 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & 0 & 1 \\ \dfrac{-a_0}{a_n} & \cdots & & \dfrac{-a_{n-2}}{a_n} & \dfrac{-a_{n-1}}{a_n} \end{bmatrix}.$$

2. Écrire le système r-réduit en x associé (2) dont la matrice $\boldsymbol{B}(x)$ est donnée ci-dessus.

3. Écrire l'équation différentielle obtenue par l'algorithme du vecteur cyclique à partir du vecteur initial $\widetilde{E} = (1, 0, \dots, 0)$ (ou $\widetilde{E} = (\underbrace{0, \dots, 0}_{kn \text{ termes}}, 1, 0, \dots)$ pour n'importe quel $k < r$) d'où l'opérateur L après normalisation éventuelle.

Remarque 2.2. Cette approche montre que l'algorithme s'applique également aux séries formelles-logarithmiques solutions mais non, en général, à celles qui contiennent des exponentielles ([L-R01] Prop 3.3).

En effet, considérons un système général $\Delta Y = 0$ et supposons que n' colonnes d'une de ses solutions fondamentales formelles s'écrivent $\widehat{H}(x)x^M$ où la matrice $\widehat{H}$ de taille $n \times n'$ est à coefficients séries formelles en x et où M est une matrice constante de taille $n' \times n'$.

La restriction de $\boldsymbol{V}$ à ses n premières lignes contient les colonnes

$$\widehat{H}(\omega^j x)(\omega^j x)^M = \widehat{H}(\omega^j x)x^M\omega^{jM} \quad \text{pour} \quad j = 0, 1, \dots, r-1.$$

Ainsi l'espace vectoriel qu'elles engendrent contient toutes les combinaisons linéaires des colonnes $\widehat{H}(\omega^j x)x^M$ pour tout j et en particulier, si l'on note comme précédemment $\widehat{H}^j$ les sous-séries des termes de r en r extraites de $\widehat{H}$, les colonnes $\widehat{H}^j(x)x^M$ pour $j = 0, 1, \dots, r-1$. En revanche, si le système $\Delta Y = 0$ admet des solutions de la forme $\widehat{H}(x)x^M e^{Q(1/x)}$ alors $Q(1/(\omega^j x)) = Q(1/x)$ seulement si $Q(1/x)$ est un polynôme en $1/x^r$ ce qui n'est pas le cas en général. Ainsi, on ne peut pas en général réaliser les combinaisons linéaires souhaitées sur $\widehat{H}$.

Remarque 2.3. Nous précisons ici le point 2 de l'algorithme précédent (version 5). Lorsque l'équation de départ (1) : $Dy = 0$ est à coefficients polynomiaux son système compagnon est, en général, à coefficients fractions rationnelles en x. Pour construire la matrice $\boldsymbol{B}(x)$ il s'agit d'extraire des coefficients de la matrice compagnon A les sous-séries des termes de r en r. Deux méthodes naturelles se présentent pour extraire les sous-séries des termes de r en r d'une série définie comme fraction rationnelle.

Variante 1 *:* L'une des méthodes consiste en un algorithme de division euclidienne dans l'anneau $\mathbb{C}(T)[X]$ après avoir remarqué que la décomposition cherchée d'un élément $a(x) \in \mathbb{C}(x)$ sous la forme

$$a(x) = a^0(t) + a^1(t)x + \cdots + a^{r-1}(t)x^{r-1} \quad \text{où} \quad t = x^r$$

traduit l'isomorphisme canonique entre $\mathbb{C}(x)$ et $\mathbb{C}(T)[X]/(X^r - T)$.

Notons $a = \dfrac{p}{q}$ une décomposition irréductible de a. On peut voir p et q comme des éléments de $\mathbb{C}[X]$ et donc aussi de $\mathbb{C}(T)[X]$.

Si $q \equiv 1$ la division euclidienne de $p(X)$ par $X^r - T$ dans $\mathbb{C}(T)[X]$ fournit une identité

$$p(X) = Q(T, X)(X^r - T) + R(T, X) \quad \text{avec} \ \deg_X R \leq r - 1.$$

Si $q \not\equiv 1$ on remarque que $q(X)$ et $X^r - T$ sont premiers entre eux. On applique l'algorithme d'Euclide pour établir l'identité de Bézout

$$Uq + V(X^r - T) = 1 \quad \text{avec} \ U \ \text{et} \ V \in \mathbb{C}(T)[X]$$

et on effectue la division euclidienne de pU par $X^r - T$ d'où une identité

$$p(X)U(T, X) = Q(T, X)(X^r - T) + R(T, X) \quad \text{avec} \ \deg_X R \leq r - 1.$$

Dans les deux cas $R(t, x)$ fournit la décomposition de $a(x)$ cherchée.

Variante 2 *:*

Une autre méthode s'appuie sur la remarque suivante.

Le système $\dfrac{d\boldsymbol{X}}{dx} = \boldsymbol{B}(x)\boldsymbol{X}$ admet la solution fondamentale $U_n(\omega)\big(\bigoplus_{j=0}^{r-1} \mathcal{Y}(\omega^j x)\big)$ où

$$U_n(\omega) = \begin{bmatrix} I_n & I_n & \cdots & I_n \\ I_n & \omega I_n & \cdots & \omega^{r-1} I_n \\ \vdots & \vdots & & \vdots \\ I_n & \omega^{r-1} I_n & \cdots & \omega^{(r-1)(r-1)} I_n \end{bmatrix}$$ est une matrice de Van der Monde par blocs construite sur $I_n, \omega I_n, \omega^2 I_n, \ldots, \omega^{r-1} I_n$. On passe donc du système $\widetilde{\Delta}\widetilde{Y} \equiv \dfrac{d\widetilde{Y}}{dx} - \dfrac{1}{x^{r+1}}\widetilde{A}(x)\widetilde{Y} = 0$ où $\widetilde{A}(x) = \bigoplus_{j=0}^{r-1} A(\omega_j x)$ au système $\dfrac{d\boldsymbol{X}}{dx} = \boldsymbol{B}(x)\boldsymbol{X}$ par la transformation de jauge constante $\widetilde{Y} = U_n(\omega)\boldsymbol{X}$. D'où, un possible calcul de $\boldsymbol{B}(x)$

par la formule

$$\begin{aligned}\boldsymbol{B}(x) &= U_n(\omega)^{-1}\widetilde{A}(x)U_n(\omega)\\ &= \frac{1}{r}U_n\Big(\frac{1}{\omega}\Big)\widetilde{A}(x)U_n(\omega)\end{aligned}$$

Notons que cette formule très simple nécessite d'introduire le nombre algébrique ω que l'on avait jusque là évité dans cette version 5 de l'algorithme.

3 Approche duale dans l'espace des opérateurs aux différences

Nous nous posons ici la même question qu'au paragraphe précédent mais sous sa forme duale : *Comment obtenir des équations aux différences auxquelles satisfont les suites des coefficients des séries $\hat{f}^j(x)$ à partir d'une équation aux différences à laquelle satisfait la suite des coefficients de la série $\hat{f}(x)$?*

Nous notons $\mathbb{A}_1^* = \mathbb{C}[m][\tau]$ l'algèbre des opérateurs aux différences à coefficients polynomiaux où m est la variable ($m \in \mathbb{Z}$ est l'indice des suites si l'on fait opérer $\mathbb{A}_1^*$ dans l'espace $\mathbb{C}^{\mathbb{Z}}$ des suites complexes et $m \in \mathbb{R}$ ou $\mathbb{C}$ si l'on fait opérer $\mathbb{A}_1^*$ dans l'espace des fonctions complexe de variable réelle ou complexe) et où τ est l'opérateur de décalage arrière ($\tau(u_m) = u_{m-1}$) ou l'opérateur de translation de 1 à droite ($\tau f(m) = f(m-1)$). Dans cette algèbre les relations de commutation sont engendrées par l'unique relation

$$m\tau - \tau m = \tau.$$

La transformation de Mellin $\mathcal{M} : \mathbb{A}_1 \to \mathbb{A}_1^*$ définie par

$$\mathcal{M}(x) = \tau \quad \text{et} \quad \mathcal{M}(\delta) = m.$$

établit un isomorphisme d'algèbres entre $\mathbb{A}_1$ et $\mathbb{A}_1^*$. On notera qu'en particulier, elle fait correspondre de part et d'autre les relations de commutation

$$\delta x - x\delta = x \quad \text{et} \quad m\tau - \tau m = \tau.$$

À un opérateur différentiel $D = \sum_{\substack{0\le i\le p\\ 0\le k\le n}} b_{i,k}x^i\delta^k$ d'ordre n et de degré p elle fait correspondre l'opérateur aux différences $D^* = \sum_{\substack{0\le i\le p\\ 0\le k\le n}} b_{i,k}\tau^i m^k$ d'ordre p et de degré n que l'on peut réécrire sous la forme $D^* = \sum_{0\le i\le p} c_i(m)\tau^i$ grâce à la relation de commutation. À l'équation différentielle (1) correspond ainsi, par transformation de Mellin, l'équation aux différences

$$D^* y = 0. \tag{1*}$$

Si une série $f(x) = \sum_{m\geq 0} u_m x^m$ est solution de l'équation (1) alors la suite de ses coefficients $(u_m)_{m\geq 0}$ est solution de l'équation (1^*) avec conditions initiales adéquates et réciproquement.

Les procédures de passage de D à D^* et vice-versa sont élémentaires. On en trouve des implantations dans [BCL] et dans *Gfun* ([SZ94]).

Les méthodes du paragraphe précédent se transposent dans ce contexte. Nous les évoquons rapidement.

3.1 Saturation galoisienne de l'équation aux différences

Comme $\mathbb{A}_1$, l'algèbre duale $\mathbb{A}_1^*$ se plonge de façon canonique dans l'algèbre $\mathbb{C}(m)[\tau]$ et celle-ci étant euclidienne à droite et à gauche on peut y calculer le ppcmg des opérateurs déduits de D^* par l'action itérée du générateur $\breve{\omega}^* : \tau \to \omega\tau$ du groupe de Galois $\mathfrak{S}_r^*$ de l'équation $\tau^r = \sigma$.

Algorithme : version 1*.

1. Pour $j = 0, 1, \ldots, r-1$, déterminer l'opérateur D_j^* en substituant $\omega^j \tau$ à τ dans l'opérateur aux différences D^*.
2. Déterminer un plus petit commun multiple à gauche dans l'algèbre $\mathbb{C}(m)[\tau]$ des opérateurs $D_0^*, D_1^*, \ldots, D_{r-1}^*$.
3. En déduire son unique représentant primitif normalisé $\Lambda = \sum_{0\leq j\leq p} \Lambda_j(m)\tau^{rj}$ à coefficients dans $\mathbb{A}_1^*$.

On peut ici encore utiliser la procédure **skew_gcdex** du paquetage Ore_algebra de MAPLE pour déterminer le ppcmg des opérateurs D_j^* (point 2.).

3.2 Élimination polynomiale

• Première variante :

On peut comme dans l'algorithme, version 2, traduire l'action du groupe de Galois par l'introduction d'une indéterminée Ω et réaliser la saturation galoisienne en éliminant Ω par un calcul de pgcd à droite dans l'algèbre $F_g^*[\Omega]$ euclidienne à droite et à gauche.

Algorithme : version 2*.

1. Déterminer le polynôme $P^* \in \mathbb{A}_1^*[\Omega]$ en substituant $\Omega\tau$ à τ dans l'opérateur D^*.
2. Par un algorithme d'Euclide dans l'algèbre $F_g^*[\Omega]$ déterminer la relation de Bézout

$$S^*(\Omega)P^*(\Omega) + T^*(\Omega)(\Omega^r - 1) = v\Lambda$$

dans laquelle S^* et de T^* appartiennent à $\mathbb{A}_1^*[\Omega]$, leurs coefficients sont des polynômes de $\mathbb{A}_1^*$ dont les contenus sont premiers entre eux, $v \in \mathbb{C}[m]$ et Λ est normalisé.

• Deuxième variante : produit de Hadamard.

Rappelons que le produit de Hadamard de deux séries est le produit terme à terme de leurs coefficients. Les sous-séries $\hat{f}^j(x) = \sum_{m\geq 0} u_{j+mr} x^{j+mr}$ des termes de r en r d'une série $\hat{f}(x) = \sum_{m\geq 0} u_m x^m$ sont donc les produits de Hadamard de $\hat{f}$ par les séries $\frac{x^j}{1-x^r} = \sum_{m\geq 0} x^{j+mr}$.

Des algorithmes généraux de calcul de produits de Hadamard par élimination existent. On trouvera par exemple, dans le paquetage *Gfun* ([SZ94] de MAPLE, une procédure **hadamardproduct** qui, étant données deux relations de récurrence vérifiées par des séries $\hat{f}$ et $\hat{g}$ respectivement, restitue une relation de récurrence vérifiée par le produit de Hadamard $\hat{f} \odot \hat{g}$ de ces séries. Dans le cas particulier qui nous occupe on peut en donner une version spécifique ; il s'agit, dans ce cas, de construire un multiple à gauche de l'opérateur aux différences $D^* \in \mathbb{A}^*$ qui s'exprime en fonction des puissances de τ^r. Pour cela, on peut observer que le quotient de $\mathbb{C}(m)[\delta]$ par D^* à droite est un $\mathbb{C}(m)$-espace vectoriel de dimension finie p, de base $(\tau^j)_{0\leq j\leq p-1}$, et chercher une relation de dépendance $\mathbb{C}(m)$-linéaire entre les représentants des puissances successives de τ^r. Dans tous les cas, il suffit d'aller jusqu'à τ^{pr}.

Algorithme : version 3*.

1. Remplacer D^* par l'opérateur unitaire $R^* = \sum_{0\leq i\leq p} \frac{b_i(m)}{b_p(m)} \tau^i \in \mathbb{C}(m)[\tau]$.
2. Pour $j \in \mathbb{N}$ variant de 0 à p (au plus) déterminer l'unique représentant $V_j \in \mathbb{C}(m)[\tau]$ de degré $\leq p-1$ de τ^{jr} modulo R^* à droite.
3. Par un algorithme de Gauss dans le $\mathbb{C}(m)$-espace vectoriel $\mathbb{C}(m)[\delta]/(R^*)$ déterminer, en supprimant les dénominateurs, la première relation
$$\sum_{0\leq j\leq p} \Lambda_j(m) V_j = 0$$
de dépendance linéaire entre les V_j à coefficients $\Lambda_j \in \mathbb{C}[m]$.
4. $\Lambda = \sum_{0\leq j\leq p} \Lambda_j(m)\tau^{rj}$ est, après normalisation éventuelle, l'opérateur aux différences cherché.

3.3 Saturation galoisienne du système compagnon

Un opérateur aux différences admet une forme compagnon analogue à celle d'une équation différentielle. Sous l'action du générateur $\breve{\omega}^* : \tau \to \omega\tau$ du groupe de galois $\mathfrak{S}_r^*$ un système $(\tau - C^*(m))Y = 0$ devient $(\tau - \omega^{-1}C^*(m))Y = 0$ et une solution $(u_m)_m$ devient $(\omega^{-m}u_m)_m$. La version qui suit est un analogue de la version 4. On note

$E_1 = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$ le vecteur-ligne représentant la première forme coordonnée dans $\mathbb{C}^p$ et $\widetilde{E}$ le vecteur ligne $\widetilde{E} = \begin{bmatrix} E_1 & E_1 & \cdots & E_1 \end{bmatrix}$ de dimension rp.

Algorithme : version 4*.

1. Écrire le système compagnon de l'équation (1*), à savoir,

$$\Delta^* Y \equiv \tau Y - C^*(m) Y = 0$$

$$\text{où } C^*(m) = \begin{bmatrix} 0 & 1 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & 0 & 1 \\ \dfrac{-c_0}{c_p} & \cdots & & \dfrac{-c_{p-2}}{c_p} & \dfrac{-c_{p-1}}{c_p} \end{bmatrix}.$$

2. Écrire le système somme directe

$$\widetilde{\Delta^*}\widetilde{Y} \equiv \tau \widetilde{Y} - \widetilde{C^*}(m)\widetilde{Y} = 0 \quad \text{où} \quad \widetilde{C^*}(m) = \bigoplus_{j=0}^{r-1} \omega^j C^*(m).$$

3. L'algorithme du vecteur cyclique appliqué au système $\widetilde{\Delta^*}\widetilde{Y} = 0$ et au vecteur $\widetilde{E}$ conduit à l'opérateur aux différences Λ.

3.4 Réduction du rang

Étant donné un système aux différences d'ordre un et de dimension p

$$\Delta^* Y \equiv (\tau - C^*(m)) Y = 0 \tag{1}$$

le système d'ordre r et de même dimension p

$$(\tau^r - \boldsymbol{B}^*(m)) Y = 0 \tag{2}$$

où

$$\boldsymbol{B}^*(m) = (\tau^{r-1} C^*(m))(\tau^{r-2} C^*(m)) \dots (\tau C^*(m)) C^*(m)$$

admet pour solutions toutes celles de $\Delta^* Y = 0$ et il est invariant sous l'action de $\mathfrak{S}_r^*$. C'est donc un analogue du système de rang réduit pour les systèmes différentiels.

Algorithme : version 5*.

1. Écrire le système compagnon de l'équation (1*), à savoir,

$$\Delta^* Y \equiv \tau Y - C^*(m) Y = 0$$

où

$$C^*(m) = \begin{bmatrix} 0 & 1 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & 0 & 1 \\ \frac{-c_0}{c_p} & \cdots & & \frac{-c_{p-2}}{c_p} & \frac{-c_{p-1}}{c_p} \end{bmatrix}.$$

2. Écrire le système r-réduit $\widetilde{\Delta^*}^r Y \equiv \tau^r Y - \boldsymbol{B}^*(m)Y = 0$ associé dont la matrice $\boldsymbol{B}^*(m)$ est donnée par

$$\boldsymbol{B}^*(m) = C^*(m-r+1)C^*(m-r+2)\dots C^*(m-1)C^*(m).$$

3. L'algorithme du vecteur cyclique appliqué au système $\widetilde{\Delta^*}^r Y = 0$ à partir du vecteur initial $\widetilde{E} = (1, 0, \dots, 0)$ conduit à l'opérateur

$$\Lambda = \sum_{0 \le j \le p} \Lambda_j(m)\tau^{rj}.$$

Remarque 3.1. On a utilisé ici un système r-réduit de même dimension p que le système initial. Comme dans le cas différentiel, on aurait pu introduire un système r-réduit de dimension pr mais il faut noter qu'on ne peut ici introduire une puissance $r^{ème}$ du décalage $\sigma = \tau^{1/r}$ comme on avait introduit la racine $t = x^{1/r}$ de la variable x. Au lieu du système (2) on utilise quelquefois le système de dimension pr

$$\tau \boldsymbol{Z} = \boldsymbol{C}^* \boldsymbol{Z} \quad \text{où} \quad \boldsymbol{C}^* = \begin{bmatrix} 0 & 0 & \dots & 0 & A \\ A & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & A & 0 \end{bmatrix} \tag{3}$$

Si la suite $(u_m)_m$ est solution de (1) la suite $\begin{bmatrix} (u_{mr})_m \\ (u_{mr-1})_m \\ \vdots \\ (u_{mr-r+1})_m \end{bmatrix}$ est solution de (3) et,

si on partitionne le vecteur inconnu $\boldsymbol{Z} = \begin{bmatrix} Z_0 \\ \vdots \\ Z_{r-1} \end{bmatrix}$ en r vecteurs de dimension p, on obtient les équations $\left\{\begin{array}{lcl} \tau Z_0 & = & AZ_{r-1} \\ \tau Z_1 & = & AZ_0 \\ & \vdots & \\ \tau Z_{r-1} & = & AZ_{r-2} \end{array}\right.$. En particulier, chaque vecteur Z_j est solution du système r-réduit (2).

4 Comparaison des diverses méthodes et exemple

4.1 Relation entre L et Λ

De façon globale, l'approche directe et l'approche duale se correspondent par l'isomorphisme de Mellin $\mathcal{M}$ qui échange les coordonnées. Cependant, les opérateurs L et Λ auxquels conduit chacune de ces deux approches ne se correspondent pas par cet isomorphisme car une permutation de l'ordre et du rôle des coordonnées n'est pas compatible avec les opérations effectuées. Ceci est lié au fait que les divisions, pgcdd, ppcmg,... sont définis dans l'algèbre $\mathbb{C}(x)[\delta]$ à coefficients rationnels alors qu'on fait agir l'isomorphisme de Mellin par permutation des coordonnées sur l'algèbre $\mathbb{A}_1 = \mathbb{C}[x][\delta]$ à coefficients polynomiaux. Les relations arithmétiques établies dans $\mathbb{C}(x)[\delta]$ induisent des relations analogues dans $\mathbb{A}_1$ en supprimant les dénominateurs. Ce faisant on remplace divisions par pseudo-divisions au sens des algèbres de Ore et on introduit des facteurs supplémentaires dont il faut tenir compte. Expliquons plus en détail :

L'opérateur différentiel L est un ppcmg dans l'algèbre $\mathbb{C}(x)[\delta]$ mais appartenant à $\mathbb{A}_1 = \mathbb{C}[x][\delta]$, des opérateurs différentiels D_j (*cf.* Algorithme : version 1). Il vérifie donc des relations de la forme

$$S_j D_j = u(x^r)L \quad \text{pour } j = 0, 1, \ldots, r-1 \tag{1}$$

où $S_j \in \mathbb{A}_1$ et où $u(x^r) \in \mathbb{C}[x^r]$.

L'opérateur aux différences Λ est un ppcmg, appartenant à $\mathbb{A}_1^*$, des opérateurs D_j^* transformés de Mellin des opérateurs D_j. Il vérifie donc des relations de la forme

$$G_j D_j^* = v(m)\Lambda \quad \text{pour } j = 0, 1, \ldots, r-1 \tag{2}$$

où $G_j \in \mathbb{A}_1^*$ et où $v(m) \in \mathbb{C}[m]$.

Par transformation de Mellin inverse, les relations (2) deviennent

$$\widetilde{G}_j D_j = v(\delta)\widetilde{\Lambda} \quad \text{pour } j = 0, 1, \ldots, r-1$$

et, en les comparant à (1), on voit que c'est $v(\delta)\widetilde{\Lambda}$ et non pas $\widetilde{\Lambda}$ lui-même en général qui est un multiple à gauche dans $\mathbb{C}(x)[\delta]$ de l'opérateur L. A fortiori, comme on le constate sur l'exemple de l'équation de Ramis–Sibuya ci-dessous, $\widetilde{\Lambda}$ n'est pas égal à

L en général. En supprimant les dénominateurs et en remarquant que $\widetilde{\Lambda}$ et L sont des polynômes en x^r et δ on en déduit une relation de la forme

$$u_1(x^r)v(\delta)\widetilde{\Lambda} = Q(x^r, \delta)L \tag{3}$$

où $u_1(x^r) \in \mathbb{C}[x^r]$, $v(\delta) \in \mathbb{C}[\delta]$ et où $Q(x, \delta) \in \mathbb{A}_1$ et la relation duale

$$u_1(\tau^r)v(m)\Lambda = Q(\tau^r, m)L^*. \tag{4}$$

À partir de la transformée de Mellin de la relation (1) , on obtient, de même, des relations du type $v_1(m)u(\tau^r)L^* = P(m, \tau^r)\Lambda$ et $v_1(\delta)u(x^r)L = P(\delta, x^r)\widetilde{\Lambda}$.

4.2 Exemple de l'équation de Ramis–Sibuya

L'équation de Ramis–Sibuya est l'équation $Dy = 0$ où l'opérateur D s'écrit

$$D = a_3(x)\partial^3 + a_2(x)\partial^2 + a_1(x)\partial + a_0(x) \tag{5}$$

avec

$$\begin{cases} a_3(x) = x^6(3x^3 - 10x^2 - 2x - 4) \\ a_2(x) = x^3(12x^5 - 47x^4 - 16x^3 - 50x^2 - 8x - 8) \\ a_1(x) = 2x(3x^6 - 14x^5 - 12x^4 - 5x^3 - 14x^2 - 6x - 4) \\ a_0(x) = 12x^4 - 14x^3 + 60x^2 + 12x + 8 \end{cases}$$

Cette équation admet pour solutions les deux exponentielles $e^{1/x}$ et e^{1/x^2} et la série de Ramis–Sibuya $\hat{f}(x) = \hat{g}(x) + \hat{g}(x^2)$ où $\hat{g}$ désigne la série d'Euler $\hat{g}(x) = \sum_{n\geq 0}(-1)^n n!\, x^{n+1}$.

Nous nous proposons d'appliquer les algorithmes précédents (directs et duaux) à cette équation.

Nous nous plaçons, pour l'instant, dans le cas $r = 2$.

L'opérateur L calculé par les algorithmes du paragraphe 2 est l'opérateur

$$\begin{aligned} L_5 = {} & x^9(3x^4 + 86x^2 + 168)\partial^5 + 2x^6(21x^6 + 691x^4 + 1598x^2 + 168)\partial^4 \\ & + x^5(162x^6 + 6061x^4 + 16226x^2 + 1848)\partial^3 \\ & + 2x^2(90x^8 + 3816x^6 + 12079x^4 + 250x^2 - 168)\partial^2 \\ & + 12x(3x^8 + 144x^6 + 642x^4 - 12x^2 + 112)\partial \\ & - 24(15x^4 + 50x^2 + 56) \end{aligned}$$

ou, si on l'exprime en fonction de l'opérateur d'Euler δ,

$$\begin{aligned} L_5 = {} & x^4\,(3\,x^4 + 86\,x^2 + 168)\,\delta^5 + 2\,x^2\,(6\,x^6 + 261\,x^4 + 758\,x^2 + 168)\,\delta^4 \\ & + x^2\,(15\,x^6 + 779\,x^4 + 2930\,x^2 - 168)\,\delta^3 \\ & + (6\,x^8 + 351\,x^6 + 2236\,x^4 - 1348\,x^2 - 336)\,\delta^2 \\ & + (-10\,x^6 + 854\,x^4 + 1036\,x^2 + 1680)\,\delta \\ & - 360\,x^4 - 1200\,x^2 - 1344. \end{aligned}$$

C'est un opérateur d'ordre 5 et non pas d'ordre maximal $nr = 6$ ce qui fournit un exemple de vecteur $\widetilde{E}$ non cyclique, comme annoncé au paragraphe 2.2.1. L'opérateur est de degré 13 si on l'exprime en fonction de ∂ mais de degré 8 seulement si on le développe suivant les puissances de δ.

L'opérateur aux différences Λ calculé au paragraphe 3 est l'opérateur

$$\begin{aligned} \Lambda_{10} = {} & 9(m-8)(m-10)^2(m-9)^2(16\,m^5 - 296\,m^4 \\ & \quad + 2058\,m^3 - 7601\,m^2 + 20271\,m - 33468)\,\tau^{10} \\ & - (1792\,m^9 - 87200\,m^8 + 1836352\,m^7 - 22004044\,m^6 \\ & \quad + 166720060\,m^5 - 847108525\,m^4 + 3019206360\,m^3 \\ & \quad - 7738806953\,m^2 + 13424517510\,m - 11698417512)(m-8)\,\tau^8 \\ & + (-1216\,m^{10} + 55392\,m^9 - 1079288\,m^8 + 11719164\,m^7 \\ & \quad - 77174132\,m^6 + 309642768\,m^5 - 659015878\,m^4 - 98207604\,m^3 \\ & \quad + 5491902246\,m^2 - 16879573428\,m + 19197313776)\,\tau^6 \\ & + (-256\,m^{10} + 7936\,m^9 - 77344\,m^8 - 103744\,m^7 + 8452288\,m^6 \\ & \quad - 81663212\,m^5 + 421470752\,m^4 - 1353600540\,m^3 \\ & \quad + 2746004600\,m^2 - 3157587760\,m + 1367558400)\,\tau^4 \\ & + (-512\,m^9 + 18944\,m^8 - 295744\,m^7 + 2553728\,m^6 \\ & \quad - 13375840\,m^5 + 43086584\,m^4 - 78262880\,m^3 \\ & \quad + 47433816\,m^2 + 81784848\,m - 130369344)\,\tau^2 \\ & + 32(m-1)(m-4)(16\,m^5 - 456\,m^4 \\ & \quad + 5066m^3 - 28333\,m^2 + 86123\,m - 126126). \end{aligned}$$

C'est un opérateur d'ordre 10 en τ (5 en τ^2) et de degré 10.

Son transformé de Mellin inverse est l'opérateur d'ordre 10 (et non pas 5 comme L_5)

$$\begin{aligned}
\widetilde{\Lambda}_{10} = {} & 16\,x^4\,(3\,x^3+10\,x^2-2\,x+4)\,(3\,x^3-10\,x^2-2\,x-4)\,\delta^{10}\\
& +8\,x^2\,(639\,x^8-5228\,x^6-2196\,x^4-288\,x^2-64)\,\delta^9\\
& +2\,x^2\,(37413\,x^8-192160\,x^6-29020\,x^4+12016\,x^2+4864)\,\delta^8\\
& +(575667\,x^{10}-1639604\,x^8+182652\,x^6+26304\,x^4-66368\,x^2+512)\,\delta^7\\
& +(2519145\,x^{10}-3228668\,x^8+1194340\,x^6\\
& \quad -201280\,x^4+190976\,x^2-17152)\,\delta^6\\
& +(6379893\,x^{10}-2007731\,x^8-1528848\,x^6\\
& \quad -943852\,x^4-118880\,x^2+237120)\,\delta^5\\
& +(9089109\,x^{10}+2101448\,x^8-2215606\,x^6\\
& \quad +2098096\,x^4-1103560\,x^2-1775584)\,\delta^4\\
& +(6651108\,x^{10}+3375593\,x^8+2148924\,x^6\\
& \quad +5161060\,x^4+5571168\,x^2+7937664)\,\delta^3\\
& +(1928556\,x^{10}+970506\,x^8+436398\,x^6\\
& \quad -19150552\,x^4-12389544\,x^2-21442336)\,\delta^2\\
& +(-234120\,x^8-545820\,x^6+33655632\,x^4+17110512\,x^2+31203904)\,\delta\\
& +162000\,x^6-19817280\,x^4-11638080\,x^2-16144128.
\end{aligned}$$

La relation (3) est vérifiée par $\widetilde{\Lambda}_{10}$ et L_5 en prenant

- $$\begin{aligned} u_1(x^2) = {} & 243\,x^{20}+34830\,x^{18}+2064960\,x^{16}+65046960\,x^{14}\\ & +1163615280\,x^{12}+11771075936\,x^{10}+65162455680\,x^8\\ & +203987266560\,x^6+362640015360\,x^4+342535495680\,x^2\\ & +133827821568,\end{aligned}$$
- $$v(\delta) = 1,$$
- $$\begin{aligned} Q(x^2,\delta) = {} & 16\,(3\,x^3-10\,x^2-2\,x-4)\\ & (3\,x^3+10\,x^2-2\,x+4)\,(3\,x^4+86\,x^2+168)^4\,\delta^5\\ & -8\,(459\,x^{10}-13002\,x^8-278300\,x^6-493880\,x^4\\ & \quad -521792\,x^2-153216)(3\,x^4+86\,x^2+168)^3\,\delta^4\end{aligned}$$

$$
\begin{aligned}
&+ 2(13365\,x^{14} - 1993644\,x^{12} - 49787448\,x^{10} \\
&\quad - 413971040\,x^{8} - 856551216\,x^{6} - 2454506176\,x^{4} \\
&\quad - 2982352128\,x^{2} - 1143862272)(3\,x^{4} + 86\,x^{2} + 168)^{2}\,\delta^{3} \\
&- (261711\,x^{18} - 38399238\,x^{16} - 826873704\,x^{14} \\
&\quad - 8277404112\,x^{12} - 71496360592\,x^{10} - 282382737504\,x^{8} \\
&\quad - 2419380097792\,x^{6} - 5180973643776\,x^{4} \\
&\quad - 5061866471424\,x^{2} - 2149514551296)(3\,x^{4} + 86\,x^{2} + 168)\,\delta^{2} \\
&+ (4644459\,x^{22} + 118731420\,x^{20} + 9162912348\,x^{18} \\
&\quad + 97916779440\,x^{16} - 4033206784368\,x^{14} - 82159591566528\,x^{12} \\
&\quad - 550884952288448\,x^{10} - 2075017277585152\,x^{8} \\
&\quad - 3910594860533760\,x^{6} - 4248923686576128\,x^{4} \\
&\quad - 2981326231683072\,x^{2} - 1097681283514368)\delta \\
&- 8084610\,x^{22} - 299382804\,x^{20} - 7863071760\,x^{18} \\
&\quad + 892227121920\,x^{16} + 27828286253280\,x^{14} \\
&\quad + 328935359770560\,x^{12} + 1797276633574912\,x^{10} \\
&\quad + 5352759561623040\,x^{8} + 8114867619102720\,x^{6} \\
&\quad + 7819726953185280\,x^{4} + 4892526662123520\,x^{2} \\
&\quad + 1607539792674816.
\end{aligned}
$$

Le transformé de Mellin L_5^* de L_5 est l'opérateur

$$
\begin{aligned}
L_5^* = {} & 3\,(m-6)\,(m-7)^2\,(m-8)^2\,\tau^8 \\
& + (m-6)\,(86\,m^4 - 1542\,m^3 + 9959\,m^2 - 26925\,m + 24632)\,\tau^6 \\
& + (168\,m^5 - 1844\,m^4 + 5554\,m^3 + 5092\,m^2 - 49450\,m + 60544)\,\tau^4 \\
& + (336\,m^4 - 2856\,m^3 + 7724\,m^2 - 6340\,m - 1944)\,\tau^2 \\
& - 336\,(m-1)\,(m-4).
\end{aligned}
$$

C'est un opérateur d'ordre 8. Il n'est donc pas multiple de l'opérateur Λ_{10} qui est d'ordre 10. De même que $\widetilde{\Lambda}_{10}$ et L_5 vérifient la relation (3), les opérateurs L_5^* et Λ_{10} vérifient la relation (4).

On trouvera, écrites en MAPLE, les procédures correspondant à chacun des algorithmes présentés ci-dessus à l'adresse électronique indiquée dans [BCL]. À ce jour, celles-ci n'ont pas été optimisées et nous n'avons pas évalué leur complexité. Leur exécution sur un certain nombre d'exemples donne une idée sans doute assez bonne de celles qui sont à retenir et à développer en priorité.

À titre indicatif, nous donnons ci-dessous les temps de calcul obtenus sur un ordinateur de type Alpha EV6.7 cadencé à 667 MHz pour chaque méthode et pour des valeurs de r allant de 2 à 10 sur l'exemple de l'équation de Ramis–Sibuya. La notation $A5(2)$ signifie «Algorithme : version 5 et variante 2». Les étoiles $*$ indiquent un temps de calcul supérieur à 1 400 s et les doubles étoiles $**$ une mémoire saturée à 128 Mb. Dans le cas de l'algorithme $A2^*$ et pour $r = 3$ la saturation de la mémoire est obtenue après un temps de calcul de 230 s seulement.

Méthodes directes :

r	2	3	4	5	6	7	8	9	10
A1	0.9	51	151	$*$					
A2	2.4	$*$							
A3	.6	2.3	3.0	9.0	8.9	28.7	21.5	196.9	50.0
A4	.4	45.2	26.0	$*$					
A5(1)	.5	1.9	2.8	9.6	9.8	41.0	28.0	231.0	71.3
A5(2)	.5	4.0	6.6	89.0	56.0	$*$			

Méthodes duales :

r	2	3	4	5	6	7	8	9	10
A1*	1.5	$**$							
A2*	2.0	$**$							
A3*	.5	2.4	2.4	12.5	10.2	43.0	30.2	123.0	71.0
A4*	.8	121.2	1050.0	$*$					
A5*(1)	.3	1.6	1.5	6.4	4.7	19.1	11.6	47.1	25.0
A5*(2)	.9	4.6	6.0	30.1	25.3	150	88	642.0	341.3

Les meilleures méthodes apparaissent être ici sans conteste celles qui utilisent la réduction du rang sans recourir à l'emploi de nombres algébriques ($A5(1)$ et sa duale $A5^*(1)$) ou celles qui reposent sur le produit de Hadamard ($A3^*$ ou sa duale $A3$). Notons que l'écart entre $A3$ et $A5(1)$ par exemple n'est pas significatif. Ainsi la procédure $A5(1)$ écrite en utilisant les procédures générales d'élimination de Gauss disponibles sous MAPLE sans tenir compte de la forme spécifique des systèmes de rang réduit n'est certainement pas optimale. À titre anecdotique nous pouvons d'ailleurs ajouter que le même algorithme $A3$ exécuté sur une machine équipée d'un processeur de type Pentium cadencé à 2,2 GHz s'est avéré deux fois plus rapide sauf dans le cas $r = 9$ où il s'est avéré 5 fois plus lent !

4.3 Remarque sur l'utilisation d'équations différentielles linéaires affines

Nous avons choisi de travailler avec des équations différentielles linéaires. On peut tout aussi bien travailler avec des équations différentielles linéaires affines («avec second membre»).

Nous nous contentons ici d'observer l'exemple de la série de Ramis–Sibuya. Celle-ci satisfait à l'équation différentielle linéaire affine

$$x^5(2-x)y'' + x^2(4+5x^2-2x^3)y' + 2(2-x+x^2)y = 4x+2x^2+10x^3-3x^4. \quad (6)$$

En adaptant les méthodes développées au paragraphe 2 on obtient les équations différentielles linéaires affines

$$\begin{aligned} L_3(y) &= 6x^6 + 56x^4 && \text{qui admet pour solution } \hat{f}^0 \\ \text{et } L_3(y) &= -x^5 - 14x^3 - 8x && \text{qui admet pour solution } \hat{f}^1 \end{aligned}$$

où L_3 est l'opérateur

$$\begin{aligned} L_3 = (x^9 + 4\,x^7)\,\partial^3 + (34\,x^6 + 8\,x^4 + 6\,x^8)\,\partial^2 \\ + (12\,x^3 + 6\,x^7 + 43\,x^5)\,\partial - 10\,x^2 - 8. \end{aligned}$$

(Une base de solutions de l'équation $L_3(y) = 0$ est donnée par $e^{1/x}$, $e^{-1/x}$ et e^{1/x^2}).

En dérivant respectivement $(6x^6 + 56x^4)^{-1}L_3(y)$ et $(-x^5 - 14x^3 - 8x)^{-1}L_3(y)$ on obtient les équations différentielles linéaires («sans second membre»)

$$L_4'(y) = 0 \quad \text{qui admet pour solution } \hat{f}^0$$

et

$$L_4''(y) = 0 \quad \text{qui admet pour solution } \hat{f}^1$$

où

$$\begin{aligned} L_4'(y) = {} & x^8\,(3\,x^2 + 28)\,\partial^4 + x^5\,(27\,x^4 + 314\,x^2 + 56)\,\partial^3 \\ & + 3\,x^4\,(18\,x^4 + 251\,x^2 + 28)\,\partial^2 + x\,(18\,x^6 + 303\,x^4 - 146\,x^2 - 56)\,\partial \\ & + 120\,x^2 + 224 \end{aligned}$$

et

$$\begin{aligned} L_4''(y) = {} & x^8\,(x^4 + 14\,x^2 + 8)\,\partial^4 + 2\,x^5\,(5\,x^6 + 85\,x^4 + 70\,x^2 + 8)\,\partial^3 \\ & + x^4\,(24\,x^6 + 485\,x^4 + 478\,x^2 + 72)\,\partial^2 \\ & + 2\,x\,(6\,x^8 + 144\,x^6 + 153\,x^4 + 2\,x^2 - 8)\,\partial \\ & + 30\,x^4 + 60\,x^2 + 16. \end{aligned}$$

Ces équations sont d'ordre 4 ce qui prouve, en particulier, que l'équation $L_5(y) = 0$ n'est pas l'équation d'ordre minimal vérifiée par l'une ou l'autre des sous-séries $\hat{f}^0$ ou $\hat{f}^1$. On vérifie que l'opérateur L_5 qui admet simultanément les deux séries $\hat{f}^0$ et $\hat{f}^1$ pour solutions est, en fait, le ppcmg de L'_4 et de L''_4. En remarquant que la série $\hat{f}^1$ est la sous-série des termes impairs de la série d'Euler et en appliquant la même méthode à l'équation d'Euler $x^2y' + y = x$ on obtient pour $\hat{f}^1$ l'équation avec second membre $x^4y'' + 2x^3y' - y = -x$ ou encore, sous forme homogène, l'équation $x^5y''' + 5x^4y'' + x(4x^2 - 1)y' + y = 0$. Celle-ci est d'ordre 3 et non pas 4 ce qui montre que l'équation $L''_4 y = 0$ n'est pas l'équation d'ordre minimal vérifiée par $\hat{f}^1$.

5 Appendice : invariants formels d'une équation ou d'un système

Dans ce paragraphe, nous indiquons comment accéder de façon explicite ou algorithmique aux invariants formels des équations ou systèmes différentiels et, en particulier, à leur différents niveaux. Le cas d'une équation est beaucoup plus facile que celui d'un système : tous les invariants formels se lisent sur un nombre fini de polygones de Newton associés à l'équation moyennant la résolution d'un certain nombre d'équations algébriques qui sont génériquement de degré un.

Quitte à réduire les systèmes à une forme compagnon on peut ainsi, en théorie, calculer tous les invariants formels d'un système. Malheureusement, une telle réduction par un algorithme de vecteur cyclique conduit souvent à des équations dont les coefficients sont énormes voire incalculables en pratique. Il est donc justifié de s'attacher à la recherche d'algorithmes opérant directement sur les systèmes.

Les invariants formels d'un système se lisent sur une solution fondamentale formelle et caractérisent chaque classe formelle. Rappelons que la classification formelle des systèmes est la classification modulo transformation de jauge méromorphe formelle $Y = T(x)Z$, i.e., dans laquelle T est une matrice inversible à coefficients méromorphes formels. La notion d'équivalence formelle en termes d'équations s'énonce moins simplement. On renvoie à [Ore33] ou à [S96] Cor. 2.6.

5.1 Invariants formels des équations différentielles

Considérons, comme au paragraphe 2, une équation différentielle

$$Dy = 0 \tag{1}$$

où l'opérateur

$$D = a_n(x)\partial^n + a_{n-1}(x)\partial^{n1} + \cdots + a_1(x)\partial + a_0(x)$$

est à coefficients méromorphes convergents ou formels en $x = 0$ (il n'est pas utile ici de se restreindre au cas polynomial) et notons $v(a)$ la valuation en x d'une série méromorphe a.

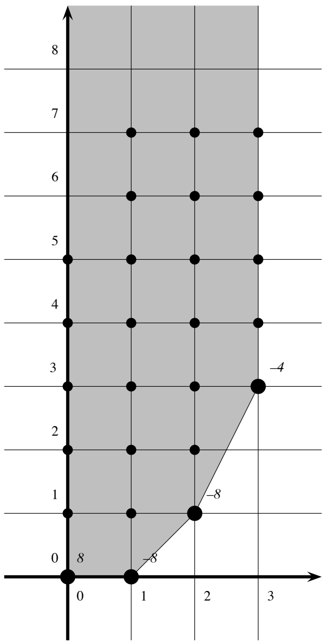
Les algorithmes de détermination des solutions formelles reposent sur les notions de polygones de Newton et d'équations caractéristiques qui leur sont attachées (voir ([BJ97] ou [M79]) par exemple). Nous esquissons ces notions ci-dessous.

• Le polygone de Newton de l'équation (1) elle-même est défini comme suit :
Dans le demi-plan $\mathbb{R}^+ \times \mathbb{R}$ on considère l'ensemble des deuxièmes quadrants

$$Q_j = \{(x, y) \in \mathbb{R}^+ \times \mathbb{R} \mid x \leq j \text{ et } y \geq v(a_j) - j\}$$

de sommets chacun des points $(j, v(a_j) - j)$, $j = 0, \ldots, n$ et $a_j \neq 0$. *Le polygone de Newton* $N(D)$ *de* D est l'enveloppe convexe de la réunion $Q = \cup_{j=0,1,\ldots,n} Q_j$ de tous ces quadrants.

La figure ci-dessous représente le polygone de Newton de l'équation de Ramis–Sibuya.



On a indiqué, en italique à côté de chaque point marqué sur le bord du polygone de Newton, le coefficient à prendre en compte dans les équations indicielle ou caractéristiques.

Polygone de Newton de l'équation de Ramis–Sibuya

Les pentes de $N(D)$ sont des nombres rationnels positifs ou nuls. Celles qui sont strictement positives sont appelées *les niveaux de* D. Quitte à utiliser une extension finie $x = t^p$ de la variable x, on peut se ramener au cas de pentes entières. Nous supposons donc désormais, pour simplifier les écritures, que les pentes de $N(D)$ sont entières.

• Les équations caractéristiques sont relatives à chaque côté de $N(D)$. Pour un côté de pente $k > 0$ on parle d'équation k-caractéristique ; pour le côté de pente nulle, s'il existe, d'équation indicielle. Ces équations sont définies par le symbole principal de D associé à la filtration par le k-poids w_k ainsi défini : on choisit $w_k(x) = \dfrac{1}{k+1}$ et $w_k(\partial) = -1$ de sorte que $w_k(x^{k+1}\partial) = 0$ et on définit le poids $w_k(D)$ comme le plus petit des k-poids de ses monômes.

Pour $k = 0$ et si le côté horizontal de $N(D)$ est de longueur $n_0 > 0$ la partie homogène de plus bas 0-poids de D est de la forme

$$\sigma_0(D) = x^m \sum_{j=0}^{n_0} a_{m+j,j} x^j \partial^j \quad \text{avec } \ a_{m+n_0,n_0} \neq 0.$$

On obtient le polynôme indiciel de D en négligeant le facteur x^m et en substituant à $x^j\partial^j$ le produit $X(X-1)\ldots(X-j+1)$ d'où *l'équation indicielle de* D :

$$\sum_{j=0}^{n_0} a_{m+j,j} X(X-1)\ldots(X-j+1) = 0$$

Pour $k > 0$ et si $N(D)$ admet un côté de pente k de longueur horizontale n_k la partie homogène de plus bas k-poids de D s'écrit (modulo des termes de k-poids supérieur si on écrit $(x^{k+1}\partial)^j$ au lieu de $x^{(k+1)j}\partial^j$)

$$\begin{aligned}\sigma_k(D) &\equiv x^{m'} \sum_{j=N'_k}^{N_k} a_{m'+(k+1)j,j} (x^{k+1}\partial)^j \\ &\equiv x^{m'} (x^{k+1}\partial)^{N'_k} \sum_{j=0}^{n_k} a_{m'+(k+1)(N'_k+j),N'_k+j} (x^{k+1}\partial)^j\end{aligned}$$

où on a noté $N'_k = n_0 + \cdots + n_{k-1}$ et $N_k = N'_k + n_k$. Ici, les deux termes extrêmes $a_{m'+(k+1)N'_k,N'_k}$ et $a_{m'+(k+1)N_k,N_k}(x^{k+1}\partial)^{n_k}$ sont non nuls. On obtient le polynôme k-caractéristique de D en négligeant le facteur $x^{m'}(x^{k+1}\partial)^{N'_k}$ et en substituant X à $x^{k+1}\partial$ d'où *l'équation k-caractéristique de D* :

$$\sum_{j=0}^{n_k} a_{m'+(k+1)(N'_k+j),N'_k+j} X^j = 0.$$

Compte tenu de la remarque précédente, cette équation n'admet pas $X = 0$ pour solution.

Décrivons à présent comment on obtient les solutions formelles de D.

• À un côté horizontal de longueur $n_0 \geq 0$ correspondent n_0 solutions formelles-logarithmiques de la forme

$$\Phi(x) = x^{\mu} \sum_{j=0}^{s-1} \varphi_j(x)(\ln x)^j$$

où $\mu \in \mathbb{C}$, $s \in \mathbb{N}^*$ et où les φ_j sont des séries formelles en x.

Celles-ci sont calculables par la méthode de Frobenius. Les différentes valeurs de μ sont les racines de l'équation indicielle comptées avec leur multiplicité. s est inférieur ou égal à la multiplicité de l'exposant μ correspondant modulo $\mathbb{Z}$. La méthode de Frobenius consiste à chercher des solutions de la forme $x^{\mu} \sum_{m\geq 0} u_m x^m$ par identification et à introduire autant de puissances successives de $\ln x$ que nécessaire pour contourner les incompatibilités. Pour les détails on renvoie à [Ince] Chap. XVI.

• À un côté de pente $k > 0$ et de longueur horizontale n_k correspondent n_k solutions de la forme

$$y = \Phi(x)e^{q(1/x)}$$

où Φ est une combinaison linéaire de séries formelles-logarithmiques du type précédent et où q est de la forme $q\left(\frac{1}{x}\right) = \frac{-\alpha_k}{kx^k} + q_1\left(\frac{1}{x}\right)$, le terme $q_1\left(\frac{1}{x}\right)$ étant un polynôme en $\frac{1}{x}$ ou en une puissance fractionnaire de $\frac{1}{x}$, de degré en $\frac{1}{x}$ strictement inférieur à k. Les différents α_k qui apparaissent sont les racines de l'équation k-caractéristique de D comptées avec leur multiplicité.

Pour obtenir les termes suivants de q il suffit d'opérer le changement d'inconnue $y = e^{-\alpha_k/kx^k} z$ et de considérer l'équation différentielle en z. Dans le polygone de Newton de cette équation on ne prend en compte que les pentes inférieures à k. On construit ainsi une suite finie d'opérateurs jusqu'à l'obtention d'une pente nulle comme seule pente admissible. À cette dernière, si elle existe, on applique la méthode de Frobenius pour déterminer le facteur Φ.

Les invariants méromorphes formels de (1) sont les triplets $(\mu \bmod \mathbb{Z}, s, q)$.

Les différents $\mu \bmod \mathbb{Z}$ sont les exposants de monodromie formelle. Les nombres entiers s associés à μ sont les dimensions des blocs de Jordan de la monodromie formelle pour la valeur propre $e^{2\pi i\mu}$. La partie exponentielle $e^{q(1/x)}$ des solutions est aussi appelée leur partie irrégulière. Dans le cas de l'équation de Ramis–Sibuya les invariants méromorphes formels sont ainsi les triplets $(0, 1, 0)$, $\left(0, 1, \frac{1}{x}\right)$, $\left(0, 1, \frac{1}{x^2}\right)$.

Des procédures de calcul de ces invariants sont implantées dans le code *desir2* ; on trouvera un pointeur sur *desir2* dans [BCL].

5.2 Invariants formels des systèmes différentiels

Nous considérons maintenant un système

$$\frac{dY}{dx} - A(x)Y = 0 \tag{2}$$

de rang de Poincaré r et nous nous proposons de déterminer les invariants de ce système sans passer par l'intermédiaire d'une forme compagnon. Naturellement, dans le cas où l'on dispose d'une forme compagnon, comme dans celui où une telle forme compagnon est facile à obtenir, il est préférable de déterminer les invariants formels sur l'équation associée.

En écrivant $A(x) = \frac{1}{x^{r+1}} \sum_{m \geq 0} A_m x^m$, nous supposons $A_0 \neq 0$ sinon r ne serait pas le rang de Poincaré et nous supposons $r \geq 0$ sinon l'origine ne serait pas un point singulier de (2), toutes les solutions seraient analytiques à l'origine et il n'y aurait pas d'invariant à déterminer.

Rappelons qu'un tel système admet une solution fondamentale formelle de la forme

$$\widehat{Y}(x) = \widehat{F}(x) x^L e^{Q(1/x)} \tag{3}$$

où $Q(1/x)$ est une matrice diagonale de polynômes en $\dfrac{1}{x}$ ou en une puissance fractionnaire de $\dfrac{1}{x}$, L est une matrice constante complexe et $\widehat{F}$ une matrice de séries méromorphes formelles ([BJL79], [L-R01] Theorem 2.2).

Les invariants formels du système (2) sont donnés par $Q(1/x)$ et L, plus précisément, par la forme de Jordan de L, les valeurs propres étant prises modulo $\mathbb{Z}$.

Lorsque $Q \equiv 0$ on dit que l'origine est une *singularité régulière* : dans ce cas, en particulier, la matrice $\widehat{F}$ est convergente dès que A l'est. Lorsque $Q \not\equiv 0$ on dit que l'origine est une *singularité irrégulière* et la matrice $\widehat{F}$ est, en général, une série divergente.

Le degré κ (éventuellement fractionnaire) de $Q(1/x)$ en la variable $\dfrac{1}{x}$ est appelé *le rang de Katz* du système. On montre que le rang de Katz κ est toujours inférieur ou égal au rang de Poincaré r et que le vrai rang de Poincaré est la partie entière supérieure de κ (i.e., le plus petit nombre entier qui lui est supérieur ou égal). Ainsi, un système est à singularité régulière si et seulement si son vrai rang de Poincaré est nul. En particulier, les systèmes de rang de Poincaré 0, quelquefois appelés *systèmes de première espèce*, sont toujours singuliers réguliers. On dispose, pour ces systèmes de première espèce, de bons algorithmes de calcul d'une solution fondamentale formelle $\widehat{F}(x)x^L$ ([CL], [S86]). On peut, dans ce cas, choisir pour L une forme de Jordan de la matrice de tête A_0 de xA. En particulier, les valeurs propres de L sont celles de la matrice A_0 et l'équation indicielle est ici remplacée par l'équation caractéristique de A_0. La matrice $\widehat{F}(x)$ n'a pas de partie polaire ; la matrice $\widehat{F}(0)$ est inversible et ses colonnes forment une base des sous-espaces caractéristiques de A_0. Le reste de la matrice $\widehat{F}(x)$ peut alors être calculé par identification dans le système et on obtient ainsi des relations de récurrence pour calculer de proche en proche toutes les parties homogènes $F_j x^j$ de $F(x)$.

Pour plus de précision, si nécessaire, nous noterons $\kappa(A), r(A), \ldots$ le rang de Katz, de Poincaré, ... du système (2) : $\dfrac{dY}{dx} - A(x)Y = 0$.

Une première étape importante dans la recherche des invariants formels est la détermination du rang de Katz. Commençons par voir comment on peut décider si ce rang est nul, c'est-à-dire, si la singularité 0 est régulière en suivant une méthode due à Moser ([Mo60]). Nous traiterons ensuite du cas singulier irrégulier.

5.2.1 Détermination de la régularité, rang de Moser et calcul des solutions régulières. La méthode de Moser consiste à faire chuter le rang de Poincaré du système jusqu'à l'obtention du vrai rang de Poincaré et ceci au moyen de transformations de jauge polynomiales de degré au plus 2. Dans le cas où le système est singulier régulier on obtient ainsi un système de première espèce dont on sait calculer les solutions ([CL], [S86]). Pour tester si le rang de Poincaré est minimal, Moser attache au système une nouvelle quantité aujourd'hui communément appelée rang de Moser et qui est aisément lisible sur le système. Nous en rappelons la définition.

Définition 5.1. (i) Le rang de Moser du système (2) : $\dfrac{dY}{dx} - A(x)Y = 0$ est le nombre rationnel

$$m(A) = \max\left(0, r + \frac{r_0}{n}\right)$$

où r désigne le rang de Poincaré du système (2) en 0, r_0 le rang de la matrice de tête A_0 de $x^{r+1}A$ et n la dimension du système (2).

(ii) Le polynôme de Moser attaché au système (2) est le polynôme de la variable λ défini par

$$\mathcal{B}(A, \lambda) = x^{r_0} \text{dét}\left(\lambda I + \frac{A_0}{x} + A_1\right)\Big|_{x=0}$$

Comme c'est le cas du rang de Poincaré le rang de Moser n'est pas invariant par transformation de jauge méromorphe mais il admet une plus petite valeur possible $\mu(A)$ que nous appellerons *vrai rang de Moser* du système.

Clairement, 0 est une singularité régulière du système (2) si et seulement si le vrai rang de Moser $\mu(A)$ du système en 0 vérifie la condition

$$\mu(A) \leq 1.$$

Ainsi, si le rang de Moser $m(A)$ est ≤ 1 le système est singulier régulier en 0. Dans le cas où $m(A) > 1$, le théorème de Moser ci-dessous donne une condition nécessaire et suffisante, simple à vérifier, pour déterminer si $m(A)$ est égal ou non au vrai rang de Moser $\mu(A)$. Dans le cas où $m(A) \neq \mu(A)$, Moser fournit, en outre, un algorithme construisant à l'aide de transformations de jauge adéquates sur le système (2) un système $\dfrac{dY}{dx} - B(x)Y = 0$ de rang de Moser minimal $m(B) = \mu(A)$ donc de rang de Poincaré minimal.

Théorème 5.2 ([Mo60]). (i) *Le rang de Moser $m(A)$ du système (2) est égal au vrai rang de Moser $\mu(A)$ si et seulement si le polynôme de Moser $\mathcal{B}(A, \lambda)$ n'est pas identiquement nul.*

(ii) *Si $\mu(A) < m(A)$ on peut réduire le rang de Moser du système par des transformations de jauge de la forme*

$$T(x) = (P_0 + P_1 x)\Delta$$

où P_0 et P_1 sont des matrices constantes, P_0 est inversible et où Δ est une matrice diagonale dont les termes diagonaux ne prennent que les valeurs 1 ou x.

Pour plus de précisions sur l'algorithme on renvoie à [H87] et [B95]. Signalons, en outre, que l'algorithme de Moser est implanté dans la procédure **moser_reduce** du paquetage DEtools de MAPLE.

Exemple 5.3. Considérons le système $Y' = AY$ où $A = \begin{bmatrix} 4\,x^{-1} & -4 \\ 2\,x^{-2} & -3\,x^{-1} \end{bmatrix}$.

La matrice A est de la forme $A = \dfrac{1}{x^2}(A_0 + A_1 x + A_2 x^2)$ où

$$A_0 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix} \quad \text{et} \quad A_2 = \begin{bmatrix} 0 & -4 \\ 0 & 0 \end{bmatrix}$$

Le rang de Poincaré r vaut 1. La matrice A_0 est nilpotente de rang $r_0 = 1$ et rang de Moser $m(A)$ du système vaut donc $m(A) = 1 + 1/2 = 3/2 > 1$. Comme

$$x \det\left(\lambda I + \frac{A_0}{x} + A_1\right) = x \begin{vmatrix} \lambda + 4 & 0 \\ 2\,x^{-1} & \lambda - 3 \end{vmatrix} = x(\lambda + 4)(\lambda - 3)$$

le polynôme de Moser $\mathcal{B}(A, \lambda)$ est identiquement nul et le vrai rang de Moser du système est strictement inférieur à $\dfrac{3}{2}$. Comme ce rang est un nombre rationnel de dénominateur 2 on peut, ici, d'ores et déjà conclure que $\mu(A) \leq 1$, donc que le système est singulier régulier en 0.

La transformation de jauge de matrice $T = \begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix}$ réduit le rang de Moser à 1. En effet, le nouveau système s'écrit $\dfrac{dY}{dx} - B(x)Y = 0$ où

$$B = T^{-1}AT - T^{-1}T' = \frac{1}{x}\begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix}.$$

Comme c'est un système de première espèce en 0 on arrête là la procédure de réduction : le système donné est un système singulier régulier en 0 équivalent au système $\dfrac{dY}{dx} - B(x)Y = 0$ ci-dessus.

Dans cet exemple, on a pu conclure que le système est singulier régulier sans avoir à effectuer la réduction du système par des transformations de Moser mais, en général, et chaque fois que la procédure de Moser requiert plusieurs étapes, on ne peut éviter le calcul effectif de réduction du système à chaque étape.

5.2.2 Cas des singularités irrégulières. Turrittin ([T55]) et Wasow ([Was]) ont, les premiers, donné en toute généralité une matrice fondamentale de solutions formelles d'un système (2). Leur démonstration conduit à une version faible de la formule (3) dans laquelle la série $\widehat{F}$ peut contenir les puissances fractionnaires de x nécessaires à l'écriture de Q.

Dans [BJL79] W. Balser, W. B. Jurkat et D. A. Lutz montrent qu'une bonne symétrisation des formules obtenues par Turrittin et Wasow permet d'obtenir la forme fine (3) puis, dans [L-R01], on exprime cette symétrisation en termes de réduction du rang de certains systèmes simples exhibant les invariants formels. Ainsi peut-on déjà lire tous les invariants formels du système sur la forme faible de Turritin-Wasow.

La démonstration de Turrittin-Wasow est algorithmique. Elle repose sur une triple récurrence portant sur le rang et la dimension du système et sur le nombre de blocs de Jordan de A_0. Elle opère en alternance des opérations du type suivant :

– Découplage : si la matrice A_0 est somme directe de deux matrices sans valeurs propres communes alors on peut scinder de façon analogue le système par une transformation de jauge formelle ([Was] Theorem 11.1).
– Changement de vecteur inconnu de la forme $Y = e^{\frac{-\alpha}{rx^r}} Z$.
– Mise sous forme réduite de Turrittin-Wasow : la matrice de tête A_0 de $x^{r+1}A$ étant sous forme de Jordan nilpotente il s'agit de faire apparaître dans les termes suivants A_j des zéros sur toutes les lignes non nulles de A_0 ([Was] Lemma 19.2).
– Cisaillement : ce sont des transformations de jauge diagonales de la forme

$$\mathrm{diag}(1, x^g, x^{2g}, \ldots, x^{(n-1)g})$$

pour un g convenable à déterminer.
– Extension algébrique convenable $x = t^s$ de la variable.

Cette construction a le mérite d'établir la partie centrale du théorème de classification formelle. Malheureusement, dans la pratique, elle s'avère très peu efficace pour calculer les invariants d'un système donné. En particulier, la mise sous forme réduite de Turritin-Wasow est extrêmement lourde et aboutit difficilement. Or, il est en général nécessaire d'y recourir plusieurs fois.

Nous présentons ci-dessous un protocole dans lequel on s'intéresse en priorité à la détermination de la partie irrégulière d'une solution fondamentale ([B97]) ; on sait alors qu'il suffit de garder les nr premiers termes de chaque coefficient du système pour garantir la validité des résultats ([BV83], [LS85]). On n'a pas de borne semblable sur le nombre de termes permettant de calculer sans erreur la partie régulière et un nombre donné de termes dans $\widehat{F}$. Un algorithme de calcul des solutions régulières d'un système singulier irrégulier permet de contourner cette difficulté ([BP99]).

Les procédures sont implantées sous Maple dans le code *isolde*. On trouvera un pointeur vers *isolde* dans [BCL].

Protocole pour le calcul de la partie irrégulière :

1. On réduit le système à son vrai rang de Moser (*cf.* Théorème 5.2).

On note encore (2) : $\dfrac{dY}{dx} - A(x)Y = 0$ avec $A(x) = \frac{1}{x^{r+1}} \sum_{j\geq 0} A_j x^j$ le système ainsi réduit, n sa dimension, r son rang de Poincaré, $m = m(A)$ son rang de Moser, $\kappa = \kappa(A)$ son rang de Katz et r_0 le rang de A_0.

2. Si $m \leq 1$, le système est à singularité régulière : sa partie irrégulière Q est nulle.

3. Si $n = 1$ et $m > 1$, la partie irrégulière Q s'obtient par intégration terme à terme de la partie polaire de $A(x)$.

4. Si A_0 admet au moins deux valeurs propres distinctes, on scinde le système (2) par la procédure de découplage de Turrittin-Wasow et on reprend en **1** pour chacun des systèmes obtenus.

5. Si A_0 admet la seule valeur propre $\alpha \neq 0$ on effectue le changement de vecteur inconnu $Y = e^{\frac{-\alpha}{rx^r}} Z$ (qui transforme le système (2) en un système de rang de Poincaré $< r$ et de matrice de tête $A_0 - \alpha I$ nilpotente) et on revient en **1**.

6. Si la matrice A_0 est nilpotente, alors

6.1. Si $r + r_0 > n$ le rang de Katz est un nombre rationnel non entier $\kappa(A) = \dfrac{r'}{s'}$, $r - 1 < \kappa(A) < r$, donné par (*cf.* [B97] Theorem 1)

$$\kappa(A) = \max\left(0, \max_{0\leq j<n} \frac{-n + j - v(c_j)}{n - j}\right) \tag{4}$$

où les c_j sont les coefficients du polynôme caractéristique de A,

$$\text{dét}(\lambda I - A) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_0,$$

et où v désigne la valuation.

6.2. Si $r + r_0 \leq n$ on augmente la valeur de r de façon à satisfaire à la condition $r + r_0 > n$ par une extension algébrique $x \mapsto x^s$ de la variable x de degré s suffisant ; on peut alors appliquer 6.1.

6.3. On opère l'extension $x = t^{s'}$ de la variable. On obtient un nouveau système $\dfrac{dZ}{dt} - B(t)Z = 0$, où $B = \frac{1}{t^{rs'+1}} \sum_{j \geq 0} s' A_j t^{js'}$, qui est de rang de Poincaré rs' et de rang de Katz entier $r' < rs'$. (Le vrai rang de Poincaré est donc égal à r'). Après avoir réduit le système à son rang de Moser minimal on est ramené au point 4 ci-dessus.

Remarque 5.4. La formule pour le rang de Katz donnée à l'étape 6.1 étend la notion de rang de Katz d'une équation différentielle : si le système (2) est le système compagnon d'une équation différentielle, cette valeur est bien égale à la plus grande pente du polygone de Newton de celle-ci.

Exemple 5.5. Considérons le système $\dfrac{dY}{dx} = A(x)Y$ où

$$A(x) = \frac{1}{x^4} \begin{bmatrix} 0 & 0 & x & 0 \\ 1 & -x^2 & x^2 & -x^2 \\ 0 & 1 & x^2 & 0 \\ x^2 & x^2 & 0 & -x^2 \end{bmatrix}.$$

Le rang de Poincaré vaut $r = 3$.

La matrice de tête $A_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ est nilpotente de rang $r_0 = 2$.

Le polynôme de Moser $\mathcal{B}(A, \lambda) = \lambda$ n'étant pas identiquement nul, le rang de Moser $m(A) = 3 + \dfrac{2}{4} = \dfrac{7}{2}$ est minimal.

La condition $r + r_0 > n$ étant satisfaite (étape 6.1 ci-dessus) on peut calculer $\kappa(A)$ en utilisant la formule (4) : comme

$$\text{dét}(\lambda I - A(x)) = \lambda^4 + \frac{\lambda^3}{x^2} - \frac{\lambda^2}{x^6} + \frac{\left(-2\,x^7 - x^2 - x^5\right)\lambda}{x^{13}} + \frac{x^2 - 1}{x^{13}},$$

on a $v(a_3) = -2$, $v(a_2) = -6$, $v(a_1) = -11$, $v(a_0) = -13$ (v désigne la valuation) et, par suite,

$$\kappa(A) = \max\left\{0, 1, 2, \frac{8}{3}, \frac{9}{4}\right\} = \frac{8}{3}.$$

En posant $x = t^3$, on obtient le système $\dfrac{dY}{dt} = B(t)Y$ où

$$B(t) = \frac{3}{t^{10}} \begin{bmatrix} 0 & 0 & t^3 & 0 \\ 1 & -t^6 & t^6 & -t^6 \\ 0 & 1 & t^6 & 0 \\ t^6 & t^6 & 0 & -t^6 \end{bmatrix}.$$

La méthode de Moser conduit à appliquer à ce système la transformation de jauge $Y = SZ$ où $S = \mathrm{diag}(t^2, t, 1, 1)$. On obtient alors le système

$$\frac{dZ}{dt} = C(t)Z \quad \text{où} \quad C(t) = \frac{1}{t^9} \begin{bmatrix} -2\,t^8 & 0 & 3 & 0 \\ 3 & -3\,t^5 - t^8 & 3\,t^4 & -3\,t^4 \\ 0 & 3 & 3\,t^5 & 0 \\ 3\,t^7 & 3\,t^6 & 0 & -3\,t^5 \end{bmatrix}$$

dont le rang de Poincaré est bien égal à 8.

La matrice de tête $C_0 = \begin{bmatrix} 0 & 0 & 3 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ est non nilpotente et admet les valeurs propres distinctes $0, 3, 3j$ et $3j^2$ (où $j^3 = 1$). Le système peut donc être découplé en 4 équations scalaires.

On obtient les invariants formels $e^{1/x}$ et $\dfrac{1}{t}e^{-3/(8t^8)-1/(4t^4)}$ pour chacune des trois déterminations $t = x^{1/3}, t = jx^{1/3}$ et $t = j^2x^{1/3}$ d'une racine cubique de x. Une solution fondamentale formelle s'écrit alors sous la forme

$$\widehat{Y}(x) = \widehat{F}(x) \begin{bmatrix} e^{1/x} & 0 \\ 0 & x^J U e^{Q(1/x)} \end{bmatrix}$$

où $\widehat{F}(x)$ est une série méromorphe formelle en x,

$$J = -\frac{1}{3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 2/3 \end{bmatrix},$$

$$U = \begin{bmatrix} 1 & 1 & 1 \\ 1 & j & j^2 \\ 1 & j^2 & j^4 \end{bmatrix}$$

et

$$Q(1/x) = \begin{bmatrix} q(1/x) & 0 & \\ 0 & q(1/(jx)) & 0 \\ 0 & 0 & q(1/(j^2x)) \end{bmatrix}$$

avec $q\left(\frac{1}{x}\right) = -\frac{3}{8x^{8/3}} - \frac{1}{4x^{4/3}}$.

5.3 Vecteur cyclique

Certains des algorithmes que nous proposons utilisent une méthode dite «de vecteur cyclique». Rappelons brièvement en quoi elle consiste et comment la mener à bien.

Étant donné un système différentiel $\partial Y = AY$ à coefficients dans un corps différentiel (K, ∂) on se propose de mettre ce système sous forme compagnon à l'aide de transformations de jauge sur K. On pourra alors identifier le système à l'équation différentielle dont il est le compagnon. Plusieurs réponses algorithmiques ont été données à ce problème, d'abord par Cope sous l'hypothèse que K soit un corps de fonctions, puis, sous des hypothèses plus générales, par P. Deligne, N. Katz,...

Il existe plusieurs variantes algorithmiques de cette méthode ; en voici une ([B93]).

On choisit $y = \Lambda Y$ où $\Lambda = [\lambda_1, \dots, \lambda_n] \in K^n$ une forme linéaire en les composantes $y_1, \dots, y_n$ de Y. Compte tenu de la relation $\partial Y = AY$ les dérivées successives de y sont, elles aussi, des formes linéaires en les composantes $y_1, \dots, y_n$ de Y soit

$$\partial^j y = \Lambda_j Y \tag{5}$$

où les matrices lignes Λ_j sont définies de proche en proche par les relations $\Lambda_j = \partial \Lambda_{j-1} + \Lambda_{j-1} A$ à partir de $\Lambda_0 = A$.

La méthode du vecteur cyclique consiste à chercher la relation linéaire d'ordre minimal satisfaite par y et ses formes dérivées successives. On obtient ainsi une équation différentielle d'ordre au plus n en l'inconnue y

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = 0. \tag{6}$$

Si celle-ci est d'ordre n on dit que Λ est *un vecteur cyclique* pour le système $\partial Y = AY$.

En posant $Z = \begin{bmatrix} y \\ \partial y \\ \vdots \\ \partial^{n-1} y \end{bmatrix}$ les relations (5) ci-dessus s'écrivent $Z = PY$ où la matrice P a pour lignes successives $\Lambda_0, \Lambda_1, \dots, \Lambda_{n-1}$. Le vecteur Λ est cyclique si et seulement si la matrice P est inversible. Z vérifie alors un système $\partial Z = BZ$, où $B = \partial P.P^{-1} + PAP^{-1}$, qui n'est autre que le système compagnon de l'équation (6). Remarquons, en particulier, que l'ensemble des vecteurs cycliques est dense dans K^n.

On trouvera une implantation en MAPLE de cet algorithme, y compris lorsque le vecteur n'est pas cyclique, dans [BCL].

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Gevrey normal form of systems of differential equations with a nilpotent linear part

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Abstract. We consider the following system of differential equations (Δ):

$$\begin{cases} \dot{x} = 2y + \Delta(x, y)\, 2x \\ \dot{y} = 3x^2 + \Delta(x, y)\, 3y \end{cases}$$

where $\Delta \in \mathbb{C}\{x, y\}$, $\Delta = x + \cdots$. The above system is a *prenormal* form of a generic perturbation of the system $\dot{x} = 2y,\ \dot{y} = 3x^2$ with nilpotent linear part and with Hamiltonian $h = y^2 - x^3$. F. Loray showed that there is a formal change of coordinates that transforms the system (Δ) into a *normal form*
(Δ^*): $\begin{cases} \dot{x} = 2y + \Delta^*\, 2x \\ \dot{y} = 3x^2 + \Delta^*\, 3y \end{cases}$ where $\Delta^* = x + A_0(h)$, $h = y^2 - x^3$ and $A_0 \in h\mathbb{C}[[h]]$.

The natural question whether this normal form Δ^* and then the normalizing transformation are convergent was stated by J.-P. Ramis, R. Moussu and F. Loray.

In this paper, we give an answer: after recalling some definitions and properties of Gevrey asymptotics, we present a "pattern recognition algorithm" for Gevrey power series that shows numerically the divergence and the Gevrey character of the power series A_0 (joint work with F. Michel and M. Teisseyre). Then we sketch the proof of the summability of the normal form and of the normalizing transformation, as well as their divergence in the generic case (joint work with R. Schäfke).

Keywords. Differential equations with nilpotent linear part, normal forms, formal solutions, Gevrey power series, Gevrey asymptotic, algorithmic methods, summability, elliptic functions.

Résumé. Nous considérons le système d'équations différentielles (Δ) :

$$\begin{cases} \dot{x} = 2y + \Delta(x, y)\, 2x \\ \dot{y} = 3x^2 + \Delta(x, y)\, 3y \end{cases}$$

où $\Delta \in \mathbb{C}\{x, y\}$, $\Delta = x + \cdots$. Le système précédent est la forme *prénormale* d'une perturbation générique du système $\dot{x} = 2y,\ \dot{y} = 3x^2$ qui possède une partie linéaire nilpotente et dont le hamiltonien est $h = y^2 - x^3$. F. Loray a montré qu'il existe une substitution formelle qui transforme le système (Δ) en une *forme normale* (Δ^*) : $\begin{cases} \dot{x} = 2y + \Delta^*\, 2x \\ \dot{y} = 3x^2 + \Delta^*\, 3y \end{cases}$ où $\Delta^* = x + A_0(h)$, $h = y^2 - x^3$ et $A_0 \in h\mathbb{C}[[h]]$. J.-P. Ramis, R. Moussu and F. Loray ont alors posé la question naturelle de la convergence de la série Δ^* et de la transformation normalisante.

Dans cet article, nous donnons la réponse : après avoir rappelé quelques définitions et propriétés d'asymptotique Gevrey, nous décrivons un "protocole d'étude" des séries Gevrey qui montre numériquement la divergence et le caractère Gevrey de la série A_0 (travail en collaboration avec F. Michel et M. Teisseyre). Puis nous présentons brièvement la preuve de la sommabilité de la série A_0 et de la transformation normalisante ainsi que leur divergence dans le cas générique (travail en collaboration avec R. Schäfke).

Mots clés. Équations différentielles à partie linéaire nilpotente, formes normales, solutions formelles, séries Gevrey, asymptotique Gevrey, méthodes algorithmiques, sommabilité, fonctions elliptiques.

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1 Introduction

The systems of analytic differential equations

$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y) \end{cases} \tag{1}$$

where $f(0, 0) = g(0, 0) = 0$ were studied in detail by Briot and Bouquet, Poincaré, Dulac, and more recently by Brjuno, Ecalle, Martinet and Ramis, Pérez-Marco and Yoccoz, Takens, Il'Yashenko, Zoladek, Cerveau and Moussu, Loray ... One main point in these works is to classify such systems, that is, to find *normal forms* equivalent to (1).
In several cases, there exists a *formal* normal form, and the problem is to know whether the formal power series involved are convergent or not.

When the linear part of a system (1) is non degenerate (that is if its eigenvalues λ_1, λ_2 are both nonzero), the particular case $\frac{\lambda_1}{\lambda_2} = -\frac{n_1}{n_2}, n_1, n_2 \in \mathbb{N}^*$ is called the *resonant saddle* case. The system (1) is either analytically linearizable or formally equivalent to a unique system (in a sense to be defined):

$$((S)_{\frac{n_1}{n_2},k,\mu}) \qquad \begin{cases} \dot{x} = n_2(1 + \mu\ u^k)\ x \\ \dot{y} = -n_1(1 + (\mu - 1)\ u^k)\ y \end{cases}$$

where $u = x^{n_1}\ y^{n_2}$, $k \geq 1$, $\mu \in \mathbb{C}$.

The integers n_1, n_2, k and μ are called the *formal invariants*. Moreover, each normalizing transformation from (1) to $(S)_{\frac{n_1}{n_2},k,\mu}$ can be obtained by composing an analytical diffeomorphism by a k-summable diffeomorphism which is a divergent normalizing transformation, nevertheless summable ([MR83], see [Ram78], [Ram80] for *summability*).

In [Lo99], F. Loray studied normal forms of systems

$$(\Delta_q) \qquad \begin{cases} \dot{x} = 2y + 2x\ \Delta(x, y) \\ \dot{y} = qx^{q-1} + qy\ \Delta(x, y) \end{cases}$$

where $q \in \mathbb{N}$, $q \geq 2$ and where $\Delta(x, y)$ is a convergent power series such that $\Delta(0, 0) = 0$ if $q = 2, 3$, and with no constant term nor monomials $x, x^2, \ldots, x^{[q/2]-1}$ if $q \geq 4$. The particular case $q = 2$ corresponds[1] to the resonant saddle case mentioned above.

[1] For example, the system $\dot{x} = x$, $\dot{y} = -y$ where $\lambda_1 = 1$, $\lambda_2 = -1$ (resonant saddle case) can be written (Δ_2) $\dot{X} = 2Y$, $\dot{Y} = 2X$ by the change of coordinates $X = \frac{(x-y)}{2}$; $Y = \frac{(x+y)}{2}$.

In this paper, we present work in progress for the case $q = 3$, when the linear part of the system (Δ_3) is nilpotent. The main results obtained so far and presented here are joint work with F. Michel and M. Teisseyre for one part ([CMT01]), with R. Schäfke for the other part ([CS00], [CS03]).

We recall that F. Loray ([Lo99]) showed that there is a substitution $(x \leftarrow xU^2(x,y), y \leftarrow yU^3(x,y), dz \leftarrow U(x,y)^{-1}dz)$ where $U(x,y)$ is a convergent or a formal power series with $U(0,0) = 1$, that transforms (Δ_3)

$$(\Delta_3) \qquad \begin{cases} \dot{x} = 2y + 2x\ \Delta(x,y) \\ \dot{y} = 3x^2 + 3y\ \Delta(x,y) \end{cases}$$

where $\Delta \in \mathbb{C}\{x,y\}$, $\Delta = x + \cdots$ into a particular formal normal form (Δ^*) with $\Delta^* = x + A_0(h)$ of the same form, but with Δ replaced by Δ^*, where $A_0 \in h\mathbb{C}[[h]]$ is a power series in the Hamiltonian variable $h = y^2 - x^3$ (see Section 3 for the details). If U is a convergent (resp. formal) power series, (Δ) is said to be *analytically equivalent* (resp. *formally equivalent*) to (Δ^*).

The question whether this normal form Δ^* and then the normalizing transformation U are convergent was stated by J.-P. Ramis, R. Moussu and F. Loray.

In joint work with F. Michel and M. Teisseyre ([CMT01]), we calculated the first 30 terms of $A_0(h)$ in several examples, using the computer algebra software MAPLE V. The computations were supported by Medicis[2]. In order to study the behaviour of the coefficients of the formal power series, we used a "pattern recognition algorithm" and obtained *numerical* evidence that the power series was divergent, but Gevrey of order 6. Moreover, in the case $q = 2$, our results agreed with the theoretical results of J. Martinet and J.-P. Ramis [MR83].

In joint work with R. Schäfke, we showed theoretical results which confirmed the Gevrey character and the summability of the normal form and of the normalizing transformation and even the divergence in the generic case ([CS00], [CS03]).

A similar approach has been used earlier by different authors in the field of singular perturbations:

In 1985, the first fifty coefficients of the formal solutions of the forced Van der Pol equation were calculated in joint work with F. Diener and M. Gaëtano ([CDG89]) and the results revealed the possible Gevrey character of the formal power series. Five years later, this conjecture was proved in [Ca91]. As a consequence, the existence of so-called "canard" solutions could be established, using Borel–Laplace summation. In fact, for this problem, the exponentially closeness of two "canard" solutions suggested the Gevrey character of the formal power series and the numerical results confirmed the Gevrey character of the formal solutions. In the problem of normal forms however, there were no such indications and numerical experimentation was needed in order to state a reasonable conjecture. Now, theoretical results confirm the Gevrey character and the divergence of the power series [CS00], [CS03].

[2] Medicis is a center of resources for computer algebra, provided by the CNRS and the Ecole Polytechnique of Palaiseau. It provided a huge computer capacity that was required by this problem.

In the first part of this article, we recall some definitions and properties of Gevrey asymptotics. In the second part, we present the "pattern recognition" method for Gevrey power series and numerical results in the case $\Delta(x, y) = ax + \cdots,\ a \neq 0$ (non-degenerate case) and also the so called degenerate case $a = 0$ ([CMT01]). In the third part, we sketch the proof of the summability of the normal form and of the normalizing transformation, in the non-degenerate case, as well as their divergence in the generic case (cf. [CS00], [CS03]).

2 Gevrey asymptotic theory

In this part, we present basic facts about Gevrey theory[3]. These can be found in the appendix A of my Habilitation dissertation ([Ca99]).

The asymptotic theory was introduced by H. Poincaré [Poi81] in order to give a meaning to formal solutions of differential equations and thus to partially solve the problem of associating a function to a divergent power series: unfortunately, such a function (even if it is a solution of the differential equation) is not uniquely determined by its asymptotic expansion.

To reduce this non-unicity, G. N. Watson [Wat12] and F. Nevanlinna [Nev19] introduced the concept of *Gevrey asymptotic expansion*. More recently, in the late 1970s, J.-P. Ramis reintroduced and developed systematically Gevrey asymptotic expansions in relation with analytic ordinary differential equations in the complex domain ([Ram78], [Ram93]).

2.1 Gevrey power series

2.1.1 Definitions and properties

Definition 2.1. For any given $k,\ A > 0$, a formal power series

$$\hat{a}(x) = \sum_{m\geq 0} a_m\, x^m \in \mathbb{C}[[x]]$$

is said to be *Gevrey of order* $1/k$ *and type* A, if there exist two nonnegative numbers C and α such that

$$\forall m,\ m \geq 0, \quad |a_m| \leq C A^{m/k}\, \Gamma(\alpha + m/k). \tag{1}$$

Remark 2.1. The property above[4] is equivalent to: there exist $K > 0$, $\beta \geq 0$ such that $\forall m,\ m > 0,\ |a_m| \leq K\, m^{\beta} \left(\frac{A}{k}\right)^{m/k} (m!)^{1/k}$.

[3] The readers interested in results about Gevrey asymptotic expansions will find more details in the references listed hereafter: [Ram78], [Ram80], [MR88], [Can89], [Tou89], [Ram93], [Bal94], [Mal95], [RS96]

[4] The power series $\sum_{m\geq 0} m!\, x^m$ is Gevrey of order 1 and type 1.

The power series $\sum_{m\geq 0} \sqrt{m!}\, x^m$ is Gevrey of order 1/2 and type 2.

Definition 2.2. We denote by $\mathbb{C}[[x]]_{\frac{1}{k},A}$ the algebra of the formal power series (in $\mathbb{C}[[x]]$) which are Gevrey of order $1/k$ and type $A > 0$ and we define[5] $\mathbb{C}[[x]]_{\frac{1}{k}} = \bigcup_{A>0} \mathbb{C}[[x]]_{\frac{1}{k},A}$.

For $k = +\infty$, we have $\mathbb{C}[[x]]_{\frac{1}{k}} = \mathbb{C}\{x\}$, the algebra of convergent power series, that is, of germs of holomorphic functions on a neighbourhood of the origin.

Remark 2.2. 1) Let $k, A, B > 0$, $A < B$, then $\mathbb{C}[[x]]_{\frac{1}{k},A} \subset \mathbb{C}[[x]]_{\frac{1}{k},B}$.

2) Let $0 < k_1 < k_2$ and $A > 0$. If $\hat{a}(x)$ is a Gevrey power series of order $1/k_2$ and type A, then $\hat{a}(x)$ is a Gevrey power series of order $1/k_1$ and type $A^{\frac{k_1}{k_2}}$.

Proposition 2.1. *$(\mathbb{C}[[x]]_{\frac{1}{k},A}, +, ., \times)$ is a commutative differential sub-algebra of $\mathbb{C}[[x]]$ over $\mathbb{C}$, for the usual derivation $\frac{d}{dx}$.*

Proposition 2.2. *The power series $\sum_{m\geq 0} a_m x^m \in \mathbb{C}[[x]]$ is Gevrey of order $1/k$ and type A if and only if $\sum_{m\geq 0} a_m x^{pm}$ is Gevrey of order $\frac{1}{pk}$ and of the same type A for $p > 0$, $p \in \mathbb{N}$.*

Proposition 2.3. *Let $\Phi(u, v)$ be an analytic function in the neighbourhood of $0 \in \mathbb{C}^2$ and let $\hat{u}$, $\hat{v} \in \mathbb{C}[[x]]_{\frac{1}{k},A}$ such that $\hat{u}(0) = 0$, $\hat{v}(0) = 0$. Then $\Phi(\hat{u}, \hat{v}) \in \mathbb{C}[[x]]_{\frac{1}{k},A}$.*

Other properties, such as an implicit function theorem in $\mathbb{C}[[x]]_{\frac{1}{k}}$, can be founded in [Mal95].

2.1.2 Formal Borel transform and Formal Laplace transform. This paragraph is widely inspired by [LR95] and [Ram93]. We introduce the *formal Borel transform* ([Bor99]) which allows us to recognize Gevrey power series, and its inverse the *formal Laplace transform.*

Definition 2.3. Let $\hat{a}(x) = \sum_{m\geq 0} a_m x^m$ be a power series and $R \geq 0$ its radius of convergence. We call *formal Borel transform of order* $1/k$ of $\hat{a}$, the power series $\hat{\mathcal{B}}_k(\widehat{a})$ defined by

$$\hat{\mathcal{B}}_k(\hat{a})(\lambda) = a_0\delta + \sum_{m\geq 0} \frac{a_{m+1}}{\Gamma(1 + m/k)}\lambda^m$$

where δ is the Dirac measure at 0.

In the particular case $k = 1$, we have $\hat{\mathcal{B}}_1(\hat{a})(\lambda) = a_0\delta + \sum_{m\geq 0} \frac{a_{m+1}}{m!}\lambda^m$.

Assume now that $a_0 = 0$ (that is to say, we study $(\hat{a}(x) - a_0)$). The formal Borel transform $\hat{\mathcal{B}}_k$ links Gevrey power series of order $1/k$ to convergent power series:

Proposition 2.4. *Let $\hat{a}(x) = \sum_{m\geq 0} a_m x^m$ be a power series and $R \geq 0$ its radius of convergence. The formal Borel transform $\hat{\mathcal{B}}_k(\hat{a})(\lambda)$ of order $1/k$ of $\hat{a}$ is a convergent power series in the λ-plane if and only if $\hat{a}(x)$ is a Gevrey power series of order $1/k$.*

[5] This last set was denoted by $\mathbb{C}[[x]]_k$ or $\mathbb{C}\{x\}_{\frac{1}{k}}$ in some older papers.

Remark 2.3. If the power series $\hat{a}(x)$ is convergent, the radius of convergence of the power series $\hat{\mathcal{B}}_k(\hat{a})(\lambda)$ is infinite, for all $k > 0$.

Consequently a power series $\hat{a}(x)$ is divergent, but Gevrey of order exactly $1/k$ and type A, if and only if $\hat{\mathcal{B}}_k(\hat{a})(\lambda)$ is a convergent power series in the λ-plane which defines an analytic function with at least one singularity (*the radius of convergence r of the convergent power series is finite and non zero*). Moreover, if the type A in (1) is minimal, the modulus of the first singularity is equal to $\frac{1}{A^{1/k}}$.

Table 2.1. Main properties of $\hat{\mathcal{B}}_1$.

$\hat{a}(x) = \sum_{m\geq 1} a_m x^m$	$\hat{\mathcal{B}}_1(\hat{a})(\lambda) = \sum_{m\geq 0} \frac{a_{m+1}}{m!}\lambda^m$
x^{p+1}, (p positive integer)	$\frac{\lambda^p}{p!}$
1	δ (Dirac measure)
$x\ \hat{a}(x)$	$\int_0^\lambda \hat{\mathcal{B}}_1(\hat{a})(u)du$
$(\hat{a} \times \hat{b})(x)$	$\hat{\mathcal{B}}_1(\hat{a}) * \hat{\mathcal{B}}_1(\hat{b})(\lambda) := \int_0^\lambda \hat{\mathcal{B}}_1(\hat{a})(u)\ \hat{\mathcal{B}}_1(\hat{b})(\lambda - u)du$
$\frac{\hat{a}(x)}{x}$ where $\hat{a}(x) = \sum_{m\geq 1} a_m x^m$	$a_1\delta + \frac{d}{d\lambda}[\hat{\mathcal{B}}_1(\hat{a})(\lambda)]$
$x^2 \frac{d}{dx}(\hat{a}(x))$	$\lambda\hat{\mathcal{B}}_1(\hat{a})(\lambda)$

Definition 2.4. The *formal Laplace transform* of order $1/k$ of a power series $\hat{b}(\lambda) = \sum_{m\geq 0} b_m\lambda^m$ is the power series $\hat{\mathcal{L}}_k(\hat{b})$ defined by

$$\hat{\mathcal{L}}_k(\hat{b})(x) = \sum_{m\geq 0} b_m \Gamma(1 + m/k)\ x^{m+1}.$$

Proposition 2.5. *The* formal Laplace transformation *is the inverse operator of the formal Borel transformation. Let* $\hat{b}(\lambda) = \sum_{m\geq 0} b_m\lambda^m$ *and* $\hat{a}(x) = \sum_{m\geq 1} a_m x^m$. *Then we have*

$$\hat{\mathcal{B}}_k(\hat{\mathcal{L}}_k)(\hat{b}) = \hat{b} \ \textit{ and } \ \hat{\mathcal{L}}_k(\hat{\mathcal{B}}_k)(\hat{a}) = \hat{a}.$$

Table 2.2. Main properties of $\hat{\mathcal{L}}_1$.

$\hat{b}(\lambda) = \sum_{m\geq 0} b_m \lambda^m$	$\hat{\mathcal{L}}_1(\hat{b})(x) = \sum_{m\geq 0} b_m m!\, x^{m+1}$
1	x
λ^p, (p positive integer)	$p!\, x^{p+1}$
$\int_0^\lambda \hat{b}(u)du = \sum_{m\geq 0} \frac{b_m}{(m+1)} \lambda^{m+1}$	$x\, \hat{\mathcal{L}}_1(\hat{b})(x)$
$\frac{d}{d\lambda}\hat{b}(\lambda) = \sum_{m\geq 1} m\, b_m\, \lambda^{m-1}$	$\frac{1}{x}\hat{\mathcal{L}}_1(\hat{b})(x) - b_0$
$\lambda \times \hat{b}(\lambda)$	$x^2 \frac{d}{dx}(\hat{\mathcal{L}}_1(\hat{b})(x))$

2.2 Gevrey asymptotic expansions

To any power series $\hat{f}$ we can associate a (non unique) analytic function f such that $\hat{f}$ is the asymptotic expansion of f.

2.2.1 Definitions. We denote by $S_{r,\alpha,\beta}$, an open sector with vertex at the origin, $S_{r,\alpha,\beta} = \{x/\ \alpha < arg\ x < \beta,\ 0 < |x| < r\}$. We denote it just $S_{\alpha,\beta}$ if $r = +\infty$.

Definition 2.5. Let $S_{r,\alpha,\beta} = \{x\ /\ 0 < \|x| < r,\ \alpha < arg\ x < \beta\}$, with $r, \alpha, \beta > 0$, an open sector on the Riemann surface of log x. $S_{r',\alpha',\beta'}$ is a *subsector* of $S_{r,\alpha,\beta}$, which we denote $S_{r',\alpha',\beta'} \prec S_{r,\alpha,\beta}$, if $0 < r' < r$ and $\alpha < \alpha' < \beta' < \beta$.

Definition 2.6. Let S be an open sector in the complex plane with vertex at the origin. Let $\hat{f}(x) = \sum_{m\geq 0} b_m\, x^m \in \mathbb{C}[[x]]$ be a formal power series. Let f be a function analytic on the sector S. We say that f is asymptotic to $\hat{f}(x) = \sum_{m\geq 0} b_m\, x^m$ on the sector S in the sense of Poincaré if for any subsector S' of S and any positive integer $N \in \mathbb{N}^*$, there is a positive constant $C_{S',N}$ such that:

$$\forall\, x \in S', \quad \left| f(x) - \sum_{m=0}^{N-1} b_m x^m \right| \leq C_{S',N} |x|^N.$$

Let $\mathcal{A}(S)$ denote the space of all functions analytic on S having an asymptotic expansion in the sense of Poincaré as $x \to 0$, x in S.

As for Gevrey power series, we introduce the *asymptotic expansion with Gevrey estimates* by imposing the additional condition:

$$C_{S',N} \leq C_{S'}\, A^{N/k}\, \Gamma(\alpha + N/k)$$

where $A, \alpha, k > 0$.

Definition 2.7. Let S be an open sector with vertex at the origin. Let f be an analytic function on S, let $\hat{f}(x) = \sum_{m\geq 0} b_m x^m \in \mathbb{C}[[x]]$ be a formal power series and let $A,\ k$ be positive real numbers. We say that f *admits* $\hat{f}(x)$ *as an asymptotic expansion of Gevrey order* $1/k$ *and of type* A *as* $x \to 0$ *on* S if there is a positive constant $\alpha > 0$, and for every subsector $S' \prec S$ there is a constant $C_{S'} > 0$, such that

$$\forall\, x \in S',\ \forall N \in \mathbb{N}^*, \quad \left| f(x) - \sum_{m=0}^{N-1} b_m x^m \right| \leq\ C_{S'}\, A^{N/k}\, \Gamma(\alpha + N/k)|x|^N. \quad (2)$$

We also say that $f(x)$ is *Gevrey* $1/k$ *asymptotic of type* A *to* $\hat{f}(x)$ *on* S.

Definition 2.8. We denote by $\mathcal{A}_{\frac{1}{k},A}(S)$ the set of all functions admitting an asymptotic expansion of Gevrey order $1/k$ and type A as $x \to 0$, $x \in S$, and we define $\mathcal{A}_{\frac{1}{k}}(S) = \bigcup_{A>0} \mathcal{A}_{\frac{1}{k},A}(S)$.

2.2.2 Properties of $\mathcal{A}_{\frac{1}{k},A}(S)$.

Remark 2.4. Let $k_1 > 0$ and let S be an open sector with vertex at the origin.

i) If $f \in \mathcal{A}_{\frac{1}{k_1}}(S)$ then $f \in \mathcal{A}(S)$ with the same asymptotic expansion.

ii) Given k_2, $0 < k_1 < k_2$, and $A > 0$, if $f \in \mathcal{A}_{\frac{1}{k_2},A}(S)$ then $f \in \mathcal{A}_{\frac{1}{k_1},A}(S)$ with the same asymptotic expansion.

iii) Given $A,\ B, 0 < A < B$, if $f \in \mathcal{A}_{\frac{1}{k_1},A}(S)$ then $f \in \mathcal{A}_{\frac{1}{k_1},B}(S)$ with the same asymptotic expansion.

Proposition 2.6. *Let* $k, A > 0$ *and let* S *be an open sector with vertex at the origin. If* $f \in \mathcal{A}_{\frac{1}{k},A}(S)$ *then its asymptotic expansion* $\sum_{m\geq 0} b_m x^m$ *is Gevrey of order* $1/k$ *and of the same type* A.

From Proposition 2.6 we conclude that $\mathcal{A}_{\frac{1}{k}}(S)$ is a sub-algebra of $\mathcal{A}(S)$ as a commutative differential algebra over $\mathbb{C}$. Moreover, with the proposition above, we can define a map J_k, called the *canonical homomorphism* ([Tou89]). This map is a homomorphism of commutative differential algebras over $\mathbb{C}$.

$$\begin{aligned} J_k : \mathcal{A}_{\frac{1}{k},A}(S) &\longrightarrow \mathbb{C}[[x]]_{\frac{1}{k},A} \\ f &\longmapsto \hat{f}(x) = \sum_{m\geq 0} b_m x^m. \end{aligned} \quad (3)$$

2.2.3 Flat functions.

Definition 2.9. Let $A, k > 0$, let S be an open sector with vertex at the origin and let f an analytic function on S. The function f is said to be *Gevrey-flat of order* $1/k$ *and type* A*, as* $x \to 0$ *in* S, if f admits the trivial asymptotic expansion $0 + 0x + \cdots$ as a Gevrey asymptotic expansion of order $1/k$ and type A, that is, if there exists $\alpha > 0$ such that for any subsector S' of S, there exists a positive constant $C_{S'} > 0$ such that for all $x \in S'$ and all $N \in \mathbb{N}^*$

$$|f(x)| \leq C_{S'}\, A^{N/k}\, \Gamma(\alpha + N/k)|x|^N.$$

Proposition 2.7. *Let* $A, k > 0$ *and let* S *be an open sector with vertex at the origin. A function* f *is Gevrey-flat of order* $1/k$ *and type* A *if and only if it has an* exponential decay of level k and type A*, on every subsector* S' *of* S*, that is, if there exists* $\rho \leq 0$ *such that*

$$\forall\, S' \prec S,\ \exists\, C_{S'} > 0,\ \forall x \in S', \quad |f(x)| \leq C_{S'}|x|^{\rho}\, e^{-\frac{1}{A|x|^k}}.$$

We denote by $\mathcal{A}_{\overline{A}}^{\leq -k}(S)$ the set of all functions in $\mathcal{A}(S)$ with an exponential decay of level k and type A and we denote by $\mathcal{A}^{<0}(S)$ the set of *infinitely flat functions at the origin* (that is the set of the functions with a trivial asymptotic expansion at 0). Thus we remark[6] that

$$\mathcal{A}_{\overline{A}}^{\leq -k}(S) = \mathcal{A}_{\frac{1}{k},A}(S) \cap \mathcal{A}^{<0}(S) = \operatorname{Ker} J_k.$$

Remark 2.5. 1) Given k_1, k_2 and A, $0 < k_1 < k_2$, $A > 0$ and S an open sector, we have $\mathcal{A}_{\overline{A}}^{\leq -k_2}(S) \subset \mathcal{A}_{\overline{A}}^{\leq -k_1}(S)$.

2) Given a level $k > 0$ and A, B, $0 < A < B$, if S is an open sector then $\mathcal{A}_{\overline{A}}^{\leq -k}(S) \subset \mathcal{A}_{\overline{B}}^{\leq -k}(S)$.

2.3 Truncated Laplace transform

2.3.1 Laplace transform.

Definition 2.10. Let $\boldsymbol{a}(\lambda)$ be an analytic function in the neighbourhood of $d_\phi = \{re^{i\phi},\ r > 0\}$, $\phi \in [0, 2\pi[$. The *Laplace transform* of level k of $\boldsymbol{a}$ is the function

$$\mathcal{L}_{\phi,k}(\boldsymbol{a})(x) = k \int_{d_\phi} e^{-\lambda^k/x^k}\, \boldsymbol{a}(\lambda)\, \frac{\lambda^{k-1}}{x^{k-1}} d\lambda\ .$$

The Laplace transform $\mathcal{L}_{\phi,k}$ is the "inverse" transform of the formal Borel transform $\hat{\mathcal{B}}_k$: for any direction d_ϕ and all $k > 0$, we have $\mathcal{L}_{\phi,k}(\hat{\mathcal{B}}_k(x^{m+1})) = x^{m+1}$ for all x in S and all m in $\mathbb{N}$.

[6]Gevrey-flatness imposes Gevrey conditions on the asymptotic expansion of f: the function $f(x) = e^{-1/\sqrt{x}}$ is *flat* in 0, in the usual sense, in the half-plane $\operatorname{Re} x > 0$: its asymptotic expansion at 0 is trivial. Moreover, this function has an exponential decay of level 1/2 for $x \in \mathbb{C}\backslash\mathbb{R}_-$, that is, it is Gevrey-flat of order 2 for $x \in \mathbb{C}\backslash\mathbb{R}_-$, *but* it is *not* Gevrey-flat of order 1.

Table 2.3. Some properties of $\mathcal{L}_{\phi,1}$.

$\boldsymbol{a}(\lambda)$	$\mathcal{L}_{\phi,1}(\boldsymbol{a})(x) = \int_{d_\phi} e^{-\lambda/x}\, \boldsymbol{a}(\lambda)\, d\lambda$
$\boldsymbol{a} * \boldsymbol{b}(\lambda)$	$\mathcal{L}_{\phi,1}(\boldsymbol{a})\ .\ \mathcal{L}_{\phi,1}(\boldsymbol{b})(x)$
$\int_0^\lambda \boldsymbol{a}(u)du$	$x\ \mathcal{L}_{\phi,1}(\boldsymbol{a})(x)$
$(\frac{d}{d\lambda})\ \boldsymbol{a}(\lambda)$	$-\boldsymbol{a}(0) + \frac{1}{x}\ \mathcal{L}_{\phi,1}(\boldsymbol{a})(x)$
$\lambda\ \boldsymbol{a}(\lambda)$	$x^2\ (\frac{d}{dx})\mathcal{L}_{\phi,1}(\boldsymbol{a})(x)$

2.3.2 Truncated Laplace transform. We denote by $D_r(0)$ the open disk of center 0 and radius r and $\bar{D}_r(0)$ the closed disk.

If $\hat{a}(x)$ is a Gevrey power series of order $1/k$, the power series $\hat{\mathcal{B}}_k(\hat{a})(\lambda)$ is convergent and there exists $r > 0$ such that $\hat{\mathcal{B}}_k(\hat{a})$ defines a holomorphic function $\boldsymbol{a}(\lambda)$ on the neighbourhood of $\bar{D}_r(0)$.

If we don't know where the singularities of $\boldsymbol{a}$ are located in the λ-plane (also called *Borel plane*)[7], we introduce the *truncated Laplace transform* of level k, denoted by $\mathcal{L}_{r,\phi,k}$:

$$\mathcal{L}_{r,\phi,k}(\boldsymbol{a})(x) = k \int_{d_{\phi,r}} e^{-\lambda^k/x^k}\ \boldsymbol{a}(\lambda)\ \frac{\lambda^{k-1}}{x^{k-1}} d\lambda$$

where $d_{\phi,r}$ is the line-segment $[0, r] \subset d_\phi$. We want $\mathcal{L}_{r,\phi,k}(\boldsymbol{a})(x)$ to be defined near $x = 0$ so we must have $\phi - \frac{\pi}{2k} \leq arg\ x \leq \phi + \frac{\pi}{2k}$.

Now we consider $\mathcal{L}_{r,0,1}(\boldsymbol{a})(x) = \int_0^r e^{-\lambda/x}\ \boldsymbol{a}(\lambda)\ d\lambda$. Here $\phi = 0$ and $k = 1$; we denote by $\mathcal{L}_r$ this uncomplete transform and $M_{\boldsymbol{a}}$ the supremum of $|\boldsymbol{a}(\lambda)|$ for $\lambda \in \bar{D}_r(0)$.

The classical properties of the Laplace transform remain valid for $\mathcal{L}_r$ with exponentially small corrections ([Ca91]). Some of these properties are listed in Table 2.4 below.

[7]This is the case, for example, if the series coefficients of $\hat{a}(x)$ are only majorized.

Table 2.4. Some properties of $\mathcal{L}_r$.

$\boldsymbol{a}(\lambda)$	$\mathcal{L}_r(\boldsymbol{a})(x) = \int_0^r e^{-\lambda/x}\, \boldsymbol{a}(\lambda)\, d\lambda,\ x > 0$
1	$x - x\, e^{-r/x}$
$\boldsymbol{a} * \boldsymbol{b}(\lambda)$	$\mathcal{L}_r(\boldsymbol{a})\, .\, \mathcal{L}_r(\boldsymbol{b})(x) - E(x)$ $\lvert E(x)\rvert \le r^2\, M_{\boldsymbol{a}} M_{\boldsymbol{b}}\, e^{-r/x}\ \forall x,\ x > 0$
$\int_0^\lambda \boldsymbol{a}(u)du,\ \lambda \in \bar{D}_r(0)$	$x\, \mathcal{L}_r(\boldsymbol{a})(x) - x\, e^{-r/x} \int_0^r \boldsymbol{a}(u)du$
$(\frac{d}{d\lambda})\, \boldsymbol{a}(\lambda)$	$-\boldsymbol{a}(0) + \boldsymbol{a}(r)\, e^{-r/x} + \frac{1}{x}\, \mathcal{L}_r(\boldsymbol{a})(x)$

2.4 Properties of J_k

We have seen that the difference between two functions in $\mathcal{A}_{\frac{1}{k},A}(S)$ with the same asymptotic expansion is exponentially small of level k and type A. So we have a sequence of differential algebras: $\mathcal{A}_{\overline{A}}^{\le -k}(S) \longrightarrow \mathcal{A}_{\frac{1}{k},A}(S) \longrightarrow \mathbb{C}[[x]]_{\frac{1}{k},A}$

2.4.1 Injectivity of J_k: Watson theorem. Watson theorem [Wat12] gives conditions on the opening of the sector S so that the asymptotic expansion of $f \in \mathcal{A}_{\frac{1}{k}}(S)$ determines the function f uniquely.

Theorem 2.8 (Watson theorem). *Let $k > 0$, let S be a sector with opening $> \pi/k$ and let $f \in \mathcal{A}(S)$. If f has an exponential decay of level k in S, then $f \equiv 0$.*

Corollary 2.9. *Let S be a sector. If the opening of S is $> \pi/k$, then the homomorphism $J_k : \mathcal{A}_{\frac{1}{k},A}(S) \longrightarrow \mathbb{C}[[x]]_{\frac{1}{k},A}$ is injective.*

2.4.2 Surjectivity of J_k: Gevrey Borel–Ritt theorem.

Theorem 2.10 (Gevrey Borel–Ritt theorem). *Let $k > 0$ and $A > 0$. Let $\hat{f}$ be a Gevrey power series of order $1/k$. Let S be an open sector. If the opening of S is $\le \pi/k$ then there exists an analytic function on S having $\hat{f}$ as Gevrey asymptotic expansion of order $1/k$.*

Remark 2.6. In [Ca99], given S and a Gevrey power series $\hat{f}$ of order $1/k$ and type A, we define a new formal Borel transform and so a new truncated Laplace transform and we construct a function admitting the asymptotic expansion $\hat{f}$ of Gevrey order $1/k$. Moreover, we give the *optimal* type of this function.

Finally, the following proposition characterizes the functions in $\mathcal{A}_{\frac{1}{k}}(S)$.

Proposition 2.11 ([RS96]). *Let $k > 0$ and let S be an open sector with opening $\leq \pi/k$ bisected by d_ϕ. Let f be an analytic function on S such that $f \in \mathcal{A}(S)$. Let $\hat{f}$ be the asymptotic expansion of f (in the Poincaré sense). We suppose that $\hat{f} \in \mathbb{C}[[x]]_{\frac{1}{k},A}$ and we denote by $R > 0$ the radius of convergence of $\hat{\mathcal{B}}_k(\hat{f} - f(0))$. For any given r, $0 < r < R$, and θ, $\theta < \frac{\pi}{(2k)}$, the following conditions are equivalent:*

(i) $f \in \mathcal{A}_{\frac{1}{k}, \frac{A}{(\cos k\theta)^k}}(S)$,

(ii) $f - \mathcal{L}_{r,\phi,k}(\hat{\mathcal{B}}_k(\hat{f})) \in \mathcal{A}^{\leq -k}_{\frac{A}{(\cos k\theta)^k}}(S)$.

2.5 Cut-off asymptotic power series. Summation to the least term.

In this subsection, we introduce the notion of *cut-off asymptotic* ([RS96]) and we explain the precise equivalence between the existence of a Gevrey asymptotic expansion and the exponential precision of a "least term" cut-off procedure.

Definition 2.11. Let $k > 0$, $A > 0$ and let S be an open sector with vertex at the origin. A function f analytic on S admits $\hat{f} = \sum_{m=0}^{+\infty} b_m x^m$ as a *cut-off asymptotic of order* $1/k$ *and type* A if there is a constant $\rho \leq 0$ and if for each proper closed subsector S' of S, there is a positive constant $C_{S'}$ such that

$$\forall\, x \in S', \quad \left| f(x) - \sum_{m=0}^{[k/A|x|^k]} b_m x^m \right| \leq C_{S'} |x|^\rho \, e^{-\frac{1}{A|x|^k}}.$$

The set of all $f \in \mathcal{O}(S)$ admitting such an asymptotic power series will be denoted by $\mathcal{C}_{\frac{1}{k},A}(S)$ and $N(x) = \left[\frac{k}{A|x|^k}\right]$ will denote the index of the cut-off asymptotic.

Theorem 2.12 ([RS96]). *A function f analytic on an open sector S has a Gevrey asymptotic expansion of order $1/k$ and of type A if and only if it has a cut-off asymptotic with Gevrey estimates of order $1/k$ and type A on that sector.*

This theorem provides a computational procedure to approach a function by a known asymptotic expansion with Gevrey estimates of order $1/k$ and type A.

Definition 2.12. Let $A, k > 0$. Let $\sum_{m \geq 0} b_m x^m$ be a Gevrey power series of order $1/k$ and type A, let $N(x) = \left[\frac{k}{A|x|^k}\right]$. The *quasi-summation to the least term* of this power series is the sum $\sum_{m=0}^{N(x)} b_m x^m$.

In fact, this quasi-sum is the summation to the least term of the majorant power series whose general term is $C\ A^{m/k}\ \Gamma(\alpha + m/k)$: the index $N(x)$ of the last term considered in the truncated power series corresponds to the index of the least term of the sequence $\{C A^{m/k} \Gamma(\alpha + m/k)|x|^m\}_m$.

This justifies the Poincaré techniques of "summation to the least term", where we truncate the power series at N^* in such a way that $|b_{N^*}||x|^{N^*}$ is minimal ([Poi90]).

2.6 An introduction to k-summability

2.6.1 Borel–Laplace summation. Let us recall that if $\hat{a}(x)$ is a Gevrey power series of order $1/k$ then the power series $\hat{\mathcal{B}}_k(\hat{a})(\lambda)$ is a convergent power series and there is a disk $D_r(0)$ in the λ-plane such that $\hat{\mathcal{B}}_k(\hat{a})$ defines a holomorphic function $\boldsymbol{a}(\lambda)$ on the neighbourhood of $\bar{D}$.

If $\boldsymbol{a}(\lambda)$ has an analytic continuation along some direction $d_\phi = \{re^{i\phi},\ r > 0\}$, $\phi \in [0, 2\pi[$, we still denote by $\boldsymbol{a}(\lambda)$ this new function and we consider its Laplace transform:

$$\mathcal{L}_{\phi,k}(\boldsymbol{a})(x) = k \int_{d_\phi} e^{-\lambda^k/x^k}\ \boldsymbol{a}(\lambda)\ \frac{\lambda^{k-1}}{x^{k-1}} d\lambda\ . \tag{1}$$

With previous notations, if $|\boldsymbol{a}(\lambda)| \leq K\ \exp(\gamma|\lambda|^k)$ for all $\lambda \in d_\phi$, with $K,\ \gamma > 0$, we say that $\boldsymbol{a}$ has an *exponential growth of order at most k at infinity on* d_ϕ. The function $\mathcal{L}_{\phi,k}(\boldsymbol{a})(x)$ is then defined on the domain[8] $\{x\ /\ \gamma - \cos(k \arg\ x - k\ \phi)|x|^{-k} < 0\}$ containing 0 on its boundary, with angular measure π/k at the origin and with a "diameter" on d_ϕ of length $(1/\gamma)^{1/k}$. We call this domain the *Borel disk in the direction* d_ϕ (it is an actual disk if $k = 1$).

Definition 2.13. Given $k > 0$ and a direction d_ϕ, a power series $\hat{f} \in \mathbb{C}[[x]]$ is *Borel–Laplace summable of level k in the direction* d_ϕ if it is Gevrey of order $1/k$ and if the sum of the convergent power series $\hat{\mathcal{B}}_k(\hat{f})(\lambda)$ has an analytic continuation $\boldsymbol{f}(\lambda)$ which is holomorphic with an exponential growth of order at most k at infinity on an open sector V in the neighbourhood of d_ϕ. Under these conditions, we say[9] that

$$f(x) = \mathcal{L}_{\phi,k}(\boldsymbol{f})(x) = k \int_{d_\phi} e^{-\lambda^k/x^k}\ \boldsymbol{f}(\lambda)\ \frac{\lambda^{k-1}}{x^{k-1}} d\lambda \tag{2}$$

is the *sum of* $\hat{f}$ *in the direction* d_ϕ in the Borel–Laplace sense.

This definition is equivalent to the *k-summability* (in the sense of Ramis) explained below.

2.6.2 k-summability.

Definition 2.14 ([Ram78], [Ram80]). Given $k > 0$ and a direction d_ϕ, a formal power series $\hat{f} \in \mathbb{C}[[x]]$ is said to be *k-summable in the direction* d_ϕ if there exists a holomorphic function f on a sector S, bisected by d_ϕ, with opening $> \pi/k$ such that $\hat{f}$ is an asymptotic expansion of f with Gevrey estimates of order $1/k$ on S.

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[8] Indeed, if $\lambda = |\lambda|e^{i\phi}$ then $|e^{-\lambda^k/x^k}\ \boldsymbol{a}(\lambda)| = e^{-|\lambda|^k\ \mathrm{Re}\,(\frac{e^{ik\phi}}{x^k})}|\boldsymbol{a}(\lambda)|$.

[9] We can also define $f(x)$ with $\hat{\mathcal{B}}_1$, with the Laplace transform of level 1 and with *ramification operators* (see [LR90]).

Under these conditions, $\hat{f}$ is a Gevrey power series of order $1/k$ and the *sum* f is unique (see Watson theorem [Wat12]). We say that f is the *sum of* $\hat{f}$ *in the direction* d_ϕ in the sense of k-summability. It is easy to verify k-summability by means of Ramis–Sibuya theorem. Unfortunately this definition of k-summability is not suitable for computational methods, but it is equivalent to definition 2.13.

Proposition 2.13. *Let $k > 0$ and let d_ϕ be a direction. Let $\hat{f}(x) \in \mathbb{C}[[x]]_{\frac{1}{k}}$. The power series $\hat{f}$ is summable by the Borel–Laplace method of level k in the direction d_ϕ if and only if $\hat{f}$ is k-summable in the direction d_ϕ.*

Definition 2.15. Let k be a positive real number. A formal power series $\hat{f} \in \mathbb{C}[[x]]$ is said to be *k-summable* if it is k-summable in every direction d except for a finite number of directions. These singular directions are called *anti-Stokes lines*[10].

3 Numerical results

In this section, we present a part of joint work with F. Michel and M. Teisseyre, published in ([CMT01]).

In [CMT01], we wrote specific algorithms deduced from F. Loray theorem ([Lo99]) in the cases $q = 2$ and $q = 3$ to guess the Gevrey character of the normal forms. In the specific case $q = 3$, we have

Theorem 3.1 ([Lo99]). *Given $\Delta \in \mathbb{C}\{x, y\}$, $\Delta = x + \cdots$, there is a unique formal power series $U(x, y) = 1+\cdots$ transforming (Δ) into its (formal) normal form (Δ^*), $\Delta^* = x + A_0(h)$, where $h = y^2 - x^3$ and $A_0 \in h\mathbb{C}[[h]]$. One has*

$$(U + 2xU_x + 3yU_y)\,\Delta^* = \Delta(xU^2, yU^3) - 2yU_x - 3x^2U_y. \tag{1}$$

3.1 Pattern recognition method

The method we use to establish numerically the Gevrey character of the normal forms is the following.

We first present a pattern "recognition" algorithm for the growth rate of the coefficients. This procedure will be part of the *Gevreytiseur* described below.

Procedure. Let

$$S(X) = \sum_{n\geq 0} a_n\, X^n \in \mathbb{C}[[X]].$$

[10]These directions are called *Stokes lines* by some authors:"Half the discontinuity in form occurs on reaching the Stokes ray, and half on leaving it the other side ([Din73]).

Step 1. The divergence of the power series.

The first step consists in studying the sequence $\left(\frac{a_{n+1}}{a_n}\right)_{n\in\mathbb{N}}$ in order to determine the radius of convergence of the power series. We study the regularity and the smoothness of the graph $n \to \ln a_n$. When the radius of convergence is zero, we can study the Gevrey character of the power series.

Step 2. First step for the Gevrey character [Gev18], [Ram93].

As we have seen in the second section, if a power series is Gevrey of order $1/k$ and of type A, then its coefficients satisfy the following upper bound condition $a_n \leq C\ A^{n/k}\ \Gamma(\alpha + \frac{n}{k})$ where C and α are positive constants. We can also conjecture (cf. [CK99]) that the coefficients a_n are asymptotically equivalent to this expression:

$$a_n \sim C\ A^{n/k}\ \Gamma\left(\alpha + \frac{n}{k}\right). \tag{2}$$

The behavior of (2) is described by the following lemma:

Lemma 3.2. *Let $C > 0$, $A > 0$, $\alpha > 0$ and $X > 0$. The sequence*

$$n \to C A^{n/k}\ \Gamma\left(\alpha + \frac{n}{k}\right) |X|^n$$

is first decreasing, then non decreasing. The minimum is realized at

$$N = \left[\frac{k}{AX^k}\right] \tag{3}$$

and this minimum is $K X^{\rho} e^{-1/AX^k}$, for some $K > 0$ and $\rho = \min\left(\frac{1}{k} - \alpha + \frac{1}{2}, 0\right)$.

We also test the behavior of $a_n X^n$ for different values of X, when X is neither "too small" nor "too large". If we observe (with the values obtained numerically) the phenomenon described in Lemma 3.2, then we can find the index N which minimize $a_n X^n$. If we choose a pair of values (X_1, X_2), we can find the pair (N_1, N_2) of smallest terms. An estimation of the Gevrey order and type follows from relation (3). Finally, when the Gevrey order and type of the resulting power series are established, we can determine the constants C, A and α.

Step 3. Method of least squares.

Note that the expression (2) can also be written as $a_n \sim (n!)^s\ B^n\ n^{\beta}\ M$. The new constants are related to the previous ones by the following relations:

$$s = \frac{1}{k}; \quad B = \left(\frac{A}{k}\right)^{1/k}; \quad \beta = \alpha - \frac{1+k}{2k}; \quad M = Ck^{\frac{1}{2}-\alpha}(2\pi)^{\frac{k-1}{2k}}. \tag{4}$$

The problem is the determination of the constants s, B, β and M in order for the curve $s\ln(n!) + n\ln(B) + \beta\ln(n) + \ln(M)$ to be as close as possible to the curve of $\ln(a_n)$. The numerical method of least squares is a first way of doing it.

But before applying the method of least squares, we have to verify that
— on one hand, the curve $\ln(a_n)$ is regular, without oscillations or discontinuities,
— on the other hand, we have sufficiently many coefficients to apply the method.

Step 4. Method of spline functions:

As the functions $\ln(n!)$ and n used in the method of least squares lose (numerically) their linear independence when n increases, we calculate the constants C, A, α and k by a method using spline functions: the value $\ln(a_n)$ is approximated by a spline cubic function $\sigma(n)$ using all a_i (for details, see [CMT01]).

Step 5. Stabilization of the poles of the Borel transform by Padé approximation ([Tho90]).

3.2 The case $q = 3$

We present results of our numerical study for the power series A_0 in Theorem 3.1. For convenience, the variable $X = x^3$ is used instead of the hamiltonian variable $h = y^2 - x^3$. The *weighted degree* defined by $\deg(x^k y^l) = 2k + 3l$ is the same for these two variables.

3.2.1 The non-degenerate case $\boldsymbol{\Delta(x, y) = ax + \cdots, a \neq 0}$. Let us consider the system (Δ). We compute $A_0(x^3)$ in the two following cases:

a) The power series $\Delta(x, y)$ contains all the monomials in x and y with coefficient 1, except for the constant term.

We obtain the following result:

$$\begin{aligned}
\Delta^* = x + x^3 A_0(x^3) &= x - \tfrac{5}{8}x^3 + \tfrac{621519}{8960}x^6 - \tfrac{2211469249179}{730562560}x^9 \\
&+ \tfrac{846013046842578301941119}{35912117649408000}x^{12} - \tfrac{63909777668931941627729822186758067}{56078926436609556480000000}x^{15} \\
&+ \tfrac{18705018090014991631174644917841054127774051547270113 7}{835090001497412158491721728000000000000}x^{18} \\
&- \tfrac{13829457742202138895346721243281578007017725890134954373894536530669783981648 9}{100801319565461475210964979717461235466240000000000000000000}x^{21} \\
&+ \cdots
\end{aligned}$$

This is an alternate power series and its coefficients increase rapidly. According with our method, we write

$$A_0(x^3) = \sum_{n \geq 0} \delta_n \, x^{3n}, \quad a_n := |\delta_n|$$

and we study the power series $\sum_{n \geq 0} a_n \, X^n$, where $X = x^3$.

The series $\frac{a_{n+1}}{a_n} \to +\infty$ and we find that the radius of convergence of Δ is zero (quotient rule). Alternatively, we can plot the graph $n \to \ln(a_n)$ and check that it is regular, without oscillations or discontinuities.

The second step of the method gives the following results:

We discover the specific behavior of the series $a_n\ X^n$ and we notice the existence of a "plateau" of minimum values.

The different values of X, x and the index of the cut-off are gathered in the next table.

X	x	index N of the cut-off
10^{-6}	0.01	30
$1.25\ 10^{-7}$	0.005	42
10^{-9}	0.001	90

As $N = \left[\frac{k}{AX^k}\right]$, we deduce that $1/k$ is near to 6.

To get better estimates of k and A, we use the other steps of the procedure. In practice (for symbolic computations) we do not consider the power series $\Delta(x, y)$: we reduce it to a polynomial, truncated at the order n. We obtain a polynomial Δ^* of the same degree. If we exclude the term in x, the polynomial Δ^* contains only monomials in $X = x^3$. However, the necessary amount of calculation to reduce $\Delta(x, y)$ increases rapidly with n. Using Maple and computer means of Medicis, it was not possible to exceed[11] $n = 210$. To apply the method of least squares, we have at most 35 terms. This allows us to determine the values of $\frac{1}{k}$ and B with sufficient precision, but we have to be careful about the estimations of other constants.

The result of the method of least squares, which we applied to the series a_n for $n = 15$ to $n = 35$ is the following: $\frac{1}{k} = 5.9934$; $\ln B = -2.0278$; $\beta = -5.3831$; $\ln(M) = 12.357$. According to (4), we find

$$A = \frac{\sqrt{B}}{2} \simeq 0.1803; \quad \alpha = \beta + 1.5 \simeq -3.727; \quad C = Mk^{\alpha-\frac{1}{2}}(2)^{\frac{1-k}{2k}} \simeq 7678015.$$

Thus, $A_0(X) \in \mathbb{C}[[X]]$ seems to be a Gevrey power series of order 6 and type 0.18.

In order to check these values, we use now the method of spline functions described in the method. We find that:

$$\frac{1}{k} = 5.97132 \quad \text{and} \quad A = 0.1896.$$

b) The power series $\Delta(x, y)$ writes:

$$x + \sum_{k+l \geq 1} \alpha_{ml} x^m y^l$$

[11] We used 1 GByte of RAM, after optimization of the program in order not to overload the memory.

where we successively consider the following cases:

1) $\forall m, l\ \alpha_{ml} = \frac{1}{2}$,

2) α_{ml} is chosen randomly among the fractions $\frac{1}{10}, \ldots, \frac{19}{10}$,

3) α_{ml} is chosen randomly among the fractions $\frac{0}{10}, \frac{1}{10}, \ldots, \frac{19}{10}, \frac{20}{10}$ (so Δ contains "gaps" which correspond to zero coefficients).

After studying the series $(\frac{a_{n+1}}{a_n})$ in order to verify that the radius of convergence of Δ^* is zero and after plotting the graph $n \to \ln(a_n)$ to analyse its regularity, we apply the method of least squares. We obtain:

1) $\frac{1}{k} = 5.994054$, 2) $\frac{1}{k} = 5.993991$, 3) $\frac{1}{k} = 5.99427$.

In each case we obtain a value of $\frac{1}{k}$ very close to 6.

We apply the method of least squares to $\frac{a_n}{(n!)^2}$ and estimate the other constants. We get:

$\Delta(x, y)$	Gevrey order $1/k$	type A	α	C
$\forall m, l,\ \alpha_{ml} = 1$	6	0.18	-3.727	7678015
$\forall m, l,\ \alpha_{ml} = 1/2$	6	0.18	-3.72	4049158
random choice of α_{ml} in $\{1/20, 2/20, \ldots, 19/20\}$	6	0.18	-3.72	3153487
random choice of α_{ml} in $\{0/10, 1/10, \ldots, 20/10\}$	6	0.18	-3.72	22612640

The results of these tests enabled us to make the following

Conjecture. If the power series $\Delta(x, y)$ contains the monomial x, the reduced power series $\Delta^* = x + XA_0(X),\ X = x^3,\ A_0 \in \mathbb{C}[[X]]$, is always Gevrey of order $\frac{1}{k} = 6$ and type $A \simeq 0.18$.

c) The power series $\Delta(x, y)$ writes:

$$ax + \sum_{m+l \geq 1} x^m y^l$$

with $a \neq 0$.

For different values of a we apply the formal reduction. It is known that the change of coordinates $x = \tilde{x}/a^2,\ y = \tilde{y}/a^3$ reduces the given Δ to one with principal terms $\tilde{x}$; we treat the example in the present form, anyway, in order to assure us that our algorithms correctly predict the dependence upon a and in order to have more examples. We verify, using the quotient rule, that the power series Δ^* has a radius of convergence which is zero and that the curve $n \to \ln(a_n)$ is regular.

The result of the method of least squares is the following: $\frac{1}{k} = 2$ in very close approximation for $a = 1; \frac{1}{2}; \frac{1}{3}$.

$\Delta(x, y)$	Gevrey order $1/k$	type A	α	C
$a = 1$	6	0.18	-3.727	7678015
$a = 1/2$	6	0.09	-3.701	1.7×10^{10}
$a = 1/3$	6	0.06	-3.667	4.6×10^{13}

Thus, numerical tests allow us to propose the following conclusions:

- The Gevrey order is always the same $\frac{1}{k} = 6$.
- The Gevrey type depends on the constants a. The value A is a non decreasing function of a.

To know the dependance between a and A, we realize about ten tests, which consist in reducing the power series.

$$\Delta(x, y) = ax + \sum_{m+l \geq 1} x^m y^l$$

where a is modified for each test; we test successively all the values: 0.2; 0.4; 0.6; ...; 2.2; 2.4.

The coefficient B is always determined by the method of least squares. Then we can plot $a \to A$.

The resulting curve is clearly linear, as it should be the case taking into consideration the above change of coordinates.

3.2.2 The degenerate case $a = 0$. A second group of tests concern power series which do not contain the monomial x (degenerate cases).

We consider the two following cases:

a) The power series $\Delta(x, y)$ is of the form $bx^3 + \sum_{p+q\geq 2,\ p\neq 0,\ 2p+3q\geq 7} x^p y^q$ $(b \neq 0)$.

After final reduction, we obtain the power series

$$\Delta^* = bx^3 + xA_1(x^3).$$

b) The power series $\Delta(x, y)$ is of the form $Cx^4 + \sum_{p+q\geq 3,\ p\neq 0,\ 2p+3q\geq 9} x^p y^q$ $(C \neq 0)$

After final reduction, we obtain the power series

$$\Delta^* = Cx^4 + x^3 A_0(x^3).$$

For each case we used a specific algorithm. Contrarily to our previous tests, the curve $n \to \ln(a_n)$ here is not regular. It is difficult to estimate correctly $\frac{1}{k}$ with the least squares and the results are very different depending on whether we consider all the coefficients of Δ^* or only part of these. It is not possible then to get to a precise conclusion: this time the divergence is weak (see [CMT01]).

4 Theoretical results

In this section, we present joint work in progress with R. Schäfke ([CS00, CS03]). We study theoretically the non-degenerate case when $q = 3$.

We consider a system of differential equations denoted by (Δ):

$$(\Delta) \qquad \begin{cases} \dot{x} = 2y + 2x\ \Delta(x, y) \\ \dot{y} = 3x^2 + 3y\ \Delta(x, y) \end{cases}$$

where $\Delta \in \mathbb{C}\{x, y\}$, $\Delta = x + \cdots$, and $\dot{} = \frac{d}{dz}$.

The system (Δ) is a *prenormal* form of a generic perturbation of the system $\dot{x} = 2y$, $\dot{y} = 3x^2$ with hamiltonian $h = y^2 - x^3$ and with a nilpotent linear part. [CM88] have shown that this prenormal form can always be achieved via a convergent transformation.

F. Loray ([Lo99]) showed that (Δ) can be transformed into a system $(\tilde{\Delta})$ – of the same form, but with Δ replaced by $\tilde{\Delta}$ – by a substitution

$$(x \leftarrow xU^2(x, y), y \leftarrow yU^3(x, y), dz \leftarrow U(x, y)^{-1}dz),$$

where $U(x, y)$ is a convergent or formal power series with $U(0, 0) = 1$. The *transformation equation* connecting Δ, $\tilde{\Delta}$ and U is

$$(U + 2xU_x + 3yU_y)\ \tilde{\Delta} = \Delta(xU^2, yU^3) - 2yU_x - 3x^2U_y \tag{1}$$

where subscript "x" denotes the partial derivative ∂_x, etc. If U is a convergent (resp. formal) power series, (Δ) is said to be *analytically* (resp. *formally*) *equivalent* to $(\tilde{\Delta})$.

We ask for the "simplest" $(\tilde{\Delta})$ equivalent to a given (Δ), that is, for a *normal form* which will be denoted by (Δ^*).

Theorem 4.1 ([Lo99]). *Given $\Delta \in \mathbb{C}\{x, y\}$, $\Delta = x + \cdots$, there is a unique formal power series $U(x, y) = 1 + \cdots$ transforming (Δ) into its (formal) normal form (Δ^*), $\Delta^* = x + A_0(h)$, where $h = y^2 - x^3$ and $A_0 \in h\mathbb{C}[[h]]$. One has*

$$(U + 2xU_x + 3yU_y)\ \Delta^* = \Delta(xU^2, yU^3) - 2yU_x - 3x^2U_y. \tag{2}$$

In joint work with R. Schäfke, we show the Gevrey character and the summability of the normal form and of the normalizing transformation and their divergence in the generic case. The main result of this section is

Theorem 4.2. *With notations as above, we have the following:*

i) *The power series* $U(x, y) = \sum_{k,l} b_{kl}\ x^k y^l$ *and* $A_0(h) = \sum_{m\geq 1} A_m h^m$ *are Gevrey* 1 *in the weighted degree, that is, there exist* $K, A > 0$ *such that for all integers* k, l, m

$$|A_m| \leq K\ A^{6m}\,(6m)!, \quad |b_{kl}| \leq K\ A^{2k+3l}\,(2k+3l)!\,,$$

where $A = 0.1844 \pm 0.0001$.

ii) *The power series* $A_0(t^6)$ *is* 1*-summable if* $\arg t \neq \frac{\pi}{6}$ *modulo* $\frac{\pi}{3}$.

iii) *There exists a non-zero analytic function* $\mathcal{Q}$[12] *such that* $\mathcal{Q}(\Delta) \neq 0$ *implies that* U *and* Δ^* *are divergent, hence the above type* A *is optimal.*

It can be shown that also $U(x, y)$ is 1-summable.

A sketch of the proof is given in the following subsections.

4.1 Change of coordinates

The solution of the unperturbed system $\dot{x} = 2y$, $\dot{y} = 3x^2$, suggests the following change of variables denoted by $(\mathcal{C})$

$$\begin{cases} x = q(s)\,t^2 \\ y = q'(s)\,t^3 \end{cases} \tag{3}$$

where q is an elliptic function verifying $q'^2 = q^3 + 1$, $q(0) = \infty$. More precisely, $q = 4\mathcal{P}$ where $\mathcal{P}$ is the Weierstrass $\mathcal{P}$-function with parameters $g_2 = 0$ and $g_3 = -1/16$ ([AS64]). Note that $h = y^2 - x^3 = t^6$. Each point (x, y) outside the cusp $y^2 = x^3$ corresponds to 6 points (s, t) where s is an element inside the hexagon $\mathcal{H}$ defined by the first six zeroes (other than 0) of q. More precisely, $\mathcal{H}$ is the hexagon with vertices $\rho^j c$, $j = 0, \ldots, 5$ where $c = 2\int_0^1 (1-t^3)^{-1/2}\,dt \simeq 2.80436$, $\rho = e^{\frac{\pi}{3}i}$. With the notation

$$\begin{aligned} \tilde{U}(s,t) &= (\mathcal{C}U)(s,t) = U(q(s)t^2, q'(s)t^3) \\ D(s,t) &= (\mathcal{C}\Delta)(s,t) \\ D^*(s,t) &= (\mathcal{C}\Delta^*)(s,t) = q(s)t^2 + t^6 A_0(t^6) \end{aligned}$$

the transformation equation (2) is simplified to

$$(\tilde{U} + t\,\tilde{U}_t)\,D^* = D(s, t\tilde{U}) - 2t\,\tilde{U}_s, \tag{4}$$

where $\tilde{U}_t = \frac{\partial \tilde{U}}{\partial t}$, $\tilde{U}_s = \frac{\partial \tilde{U}}{\partial s}$. This follows from $2t\,\tilde{U}_s = \mathcal{C}(2yU_x + 3x^2U_y)(s,t)$, $t\,\tilde{U}_t = \mathcal{C}(2xU_x + 3yU_y)(s,t)$ and $D(s, t\tilde{U}) = \mathcal{C}(\Delta(xU^2, yU^3))(s,t)$.

[12] More precisely, $\mathcal{Q}$ is an analytic function of *all* the coefficients of Δ.

We are not looking for arbitrary solutions in s and t of this equation but for solutions with certain symmetry properties that can be expressed in terms of power series in x and y. For this, we introduce the following $\mathbb{C}$-vector spaces: H_m the space of homogeneous polynomials $\sum_{2k+3l=m} a_{kl}x^k y^l$ of weighted degree m and $\mathcal{E}_m$ the space of meromorphic functions on $\mathbb{C}$ that are p_0- and p_1-periodic ($p_0 = 2ia$ and $p_1 = 2i\rho a$, where $a = \int_1^\infty (t^3-1)^{-1/2}\, dt \simeq 2.42865$, are fundamental periods of q), whose only pole inside the hexagon $\mathcal{H}$ is 0, of order $\leq m$, and such that $f(\rho s) = \rho^{-m} f(s)$.

Now the coordinate change (3) induces a bijection I between H_m and $\mathcal{E}_m$ by $I(f)(s)t^m = \mathcal{C}(f)(s,t)$. Let

$$\mathcal{S} = \left\{ \sum_{m=0}^{\infty} f_m(s)t^m / f_m \in \mathcal{E}_m \right\} \quad \text{and} \quad \mathcal{S}_n = \left\{ \sum_{m=n}^{\infty} f_m(s)t^m / f_m \in \mathcal{E}_m \right\}.$$

Using the above notation, given $D \in q(s)t^2 + \mathcal{S}_6$, we are looking for a solution $\tilde{U} \in 1 + \mathcal{S}_5$, $D^* = q(s)t^2 + A_0(t^6)$ of (4). Let:

$$\begin{aligned} \tilde{U}(s,t) &= 1 + t\, W(s,t) \\ D(s,t) &= q(s)\, t^2 + t^2\, E(s,t) \\ D^*(s,t) &= q(s)\, t^2 + t^2\, E^*(s,t) \end{aligned}$$

Equation (4) becomes

$$\begin{aligned} 2W_s + t^2\, q(s) W_t + E^* = q(s)t^2 W^2 - 2tWE^* - t^2 W_t E^* \\ + E(s, t(1+tW))(1+tW)^2. \end{aligned} \tag{5}$$

We now rewrite equation (5) in the Borel plane, i.e. we apply the *formal Borel transformation with respect to t* $\hat{\mathcal{B}}$, defined by $\hat{\mathcal{B}}(t^n) = \frac{\tau^{n-1}}{(n-1)!}$ for $n \geq 1$, to all the preceding power series. Let $E(s,t(1+tW))(1+tW)^2 = \sum_{n=0}^\infty F_n(s,t)t^n\; W^n$ and

$$\begin{aligned} \boldsymbol{W}(s,\tau) &= \hat{\mathcal{B}}\left(W(s,t)\right), \\ \boldsymbol{E}^*(\tau) &= \hat{\mathcal{B}}(E^*(s,t)), \\ \boldsymbol{F}_n(s,\tau) &= \hat{\mathcal{B}}(F_n(s,t)). \end{aligned}$$

Using properties of the Borel transform, in particular $\hat{\mathcal{B}}\left(t^2 \frac{\partial}{\partial t} f(t)\right) = \tau\, \hat{\mathcal{B}}(f(t))$, equation (5) can be written as follows in the Borel plane

$$\begin{aligned} \mathcal{L}(\boldsymbol{W}, \boldsymbol{E}^*) &:= 2\boldsymbol{W}_s + q(s)\tau \boldsymbol{W} + \boldsymbol{E}^* = \mathcal{G}(\boldsymbol{W}, \boldsymbol{E}^*), \text{ where} \\ \mathcal{G}(\boldsymbol{W}, \boldsymbol{E}^*)(s,\tau) &:= q(s)\tau * \boldsymbol{W} * \boldsymbol{W} - 2\boldsymbol{W} * \boldsymbol{E}^* - (\tau \boldsymbol{W}) * \boldsymbol{E}^* \\ &\qquad + \boldsymbol{F}_0 + \sum_{n=1}^{\infty} \boldsymbol{F}_n * \tau^{n-1} * \boldsymbol{W}^{*n}/(n-1)!, \end{aligned} \tag{6}$$

where $*$ denotes the convolution product with respect to τ. The restrictions are the following: datas are $\tau^3 \boldsymbol{F}_n \in \mathcal{S}_{max(n,6)}$ and we are looking for $\boldsymbol{E}^*(\tau) \in \tau^3\mathbb{C}[[\tau^6]]$, $\tau^2 \boldsymbol{W} \in \mathcal{S}_5$.

4.2 The linearized problem

Now we consider the linearization of (6), that is, given $\tau^3 \boldsymbol{G} \in \mathcal{S}_6$, to find $\tau^2 \boldsymbol{W} \in \mathcal{S}_5$, $\boldsymbol{E}^*(\tau) \in \tau^3\mathbb{C}[[\tau^6]]$ such that

$$2\boldsymbol{W}_s + q(s)\tau \boldsymbol{W} + \boldsymbol{E}^* = \boldsymbol{G}. \tag{7}$$

This linear ordinary differential equation can be solved by "variation of constants". The "constant" of integration and $\boldsymbol{E}^*$ are uniquely determined by the conditions (in particular $\boldsymbol{W}$ has to be a single valued function of s). We find

$$\boldsymbol{W}(s,\tau) = e^{-\tau I(s)/2} \int_\infty^s e^{\tau I(\sigma)/2} (\boldsymbol{G}(\sigma,\tau) - \boldsymbol{E}^*(\tau))\, d\sigma \tag{8}$$

where I is an antiderivative of q, and

$$\boldsymbol{E}^*(\tau) = \frac{1}{\boldsymbol{R}(\tau)} \operatorname{res}(e^{\tau I(\sigma)/2} \boldsymbol{G}(\sigma,\tau)\ ,\ \sigma = 0) \quad \text{with } \boldsymbol{R}(\tau) = \operatorname{res}(e^{\tau I(\sigma)/2}\ ,\ \sigma = 0). \tag{9}$$

In (8), the path of integration goes from ∞ to s avoiding the poles of q and in such a way that $\operatorname{Re}(\tau\, I(s))$ tends to $-\infty$ as $s \to \infty$.

Since (7) has a unique solution $(\boldsymbol{W}, \boldsymbol{E}^*)$ satisfying the restrictions mentioned above, we obtain two linear operators $\mathcal{W}(\boldsymbol{G}) := \boldsymbol{W}$, $\mathcal{E}(\boldsymbol{G}) := \boldsymbol{E}^*$.

The zeroes of the function $\boldsymbol{R}$ of (9) are important since they introduce the singularities of $\boldsymbol{E}^*$ (and $\boldsymbol{W}$) in the Borel plane.

4.3 Zeroes of the function R

We study the zeroes of the function

$$\boldsymbol{R}(\tau) = \operatorname{res}(e^{\tau\, I(s)/2}, s = 0) \tag{10}$$

because they are linked to the singularities of $\boldsymbol{E}^*$ and $\boldsymbol{W}$ in the Borel plane.

We recall that $q(s)$ is an elliptic function verifying $q'^2 = q^3 + 1,\ q(0) = \infty$. It has a pole of order 2 at $s = 0$ and also at $p_j := 2ia\rho^j,\ j = 0, \ldots, 5$ where $\rho = e^{\frac{\pi i}{3}}$ and $a \simeq 2.42865$. The complex numbers $p_0, p_1, \ldots, p_5$ are periods of q. Let $c \simeq 2.80436$; the first zeroes of q except 0 are $\rho^j c$, $j = 0, \ldots, 5$.

The function $I(s)$ is the antiderivative of $q(s)$ whose Laurent power series at $s = 0$ has no constant term; $I(s) = -4\xi$ where ξ is the Weierstrass' zeta function with parameters $g_2 = 0$ and $g_3 = -1/16$ ([AS64]). I is a pseudo-periodic function. Its (simple) zeroes are $p_0/2, p_1/2, \ldots, p_5/2$ and its saddle-points are the zeroes of $I' = q$, that is, the vertices $\rho^j c,\ j = 0, \ldots, 5$ of the hexagon $\mathcal{H}$.

Note that $\boldsymbol{R}$ is an entire function. As $I(\rho s) = \rho^{-1} I(s)$, we have $\boldsymbol{R}(\rho\tau) = \rho \boldsymbol{R}(\tau)$ and the coefficients of $\boldsymbol{R}$ are real, as well as those of q and I. So

$$\boldsymbol{R}(\tau) = \tau \sum_{m=0}^{\infty} R_m \tau^{6m} \quad \text{with } R_m \in \mathbb{R},\ \forall\, m \geq 0. \tag{11}$$

We have

$$\begin{aligned}\boldsymbol{R}(\tau) = -2\,\tau - \frac{1}{12600}\tau^7 - \frac{17}{856215360000}\tau^{13} \\ - \frac{31}{88857944439264000000}\tau^{19} - \cdots .\end{aligned} \tag{12}$$

$$\boldsymbol{R}(\tau) = \tau \sum_{m=0}^{\infty} R_m \tau^{6m} \quad \text{with } R_m \in \mathbb{R},\ \forall\, m \geq 0. \tag{13}$$

If τ is a zero of $\boldsymbol{R}$, then $\rho^\nu \tau$ and $\rho^\nu \bar{\tau}$, $\nu = 0, \ldots, 5$ are zeroes of R too[13].

Lemma 4.3. *The function* $\boldsymbol{R}$ *has an exponential growth as* $|\tau| \to \infty$*; more precisely*

$$|\boldsymbol{R}(\tau)| \leq C \exp(D\,|\tau|), \quad \textit{where } C = \frac{3c}{\pi} \leq 2.68,\ D = \frac{I(-c)}{2} \leq 0.863. \tag{14}$$

Proof. The formula

$$\boldsymbol{R}(\tau) = \frac{1}{2\pi i} \int_{\mathcal{H}} e^{\tau I(s)/2}\, ds \tag{15}$$

proves the lemma.

Here is the main result of this subsection:

Theorem 4.4. *The zeroes of the function* $\boldsymbol{R}$ (*other than* $\tau = 0$) *lie on the rays* $\arg \tau = (2l+1)\frac{\pi}{6}$, $l = 0, \ldots, 5$ *and form six sequences* $m_k e^{(2l+1)\pi i/6}$, $k \in \mathbb{N}$, $l = 0, \ldots, 5$. *One has* $|m_0 - 5.4204| < 0.0002$ *and* $|m_k - t_k| < 12 t_k^{-3}$ *with* $t_k = \frac{\pi}{I(-c)}(3+4k)$ *if* $k \geq 1$*; here* $I(-c) = 2\int_0^1 t\,(1-t^3)^{-1/2}\, dt \simeq 1.72474$.

Sketch of the proof. First, we study the zeroes near the origin. We have the following:

Theorem 4.5. *The function* $\boldsymbol{R}$ *has exactly* 7 *simple zeroes in the disk* $|\tau| < 10.6$. *These are* $\tau = 0$ *and* $\tau = \rho^\nu i\, m_0$, $\nu = 0, \ldots, 5$, *where* m_0 *is real and* $|m_0 - 5.4204| <$ 0.0002.

We write

$$\boldsymbol{R}(\tau) = \boldsymbol{F}(\tau) + \boldsymbol{H}(\tau), \quad \text{where } \boldsymbol{F}(\tau) = -2\,\tau - \frac{1}{12600}\tau^7 - \frac{17}{856215360000}\tau^{13} \tag{16}$$

[13] We have more with the symmetries: $\boldsymbol{R}(\rho^\nu \tau) = \rho^\nu \boldsymbol{R}(\tau)$, $\boldsymbol{R}(\rho^\nu \bar{\tau}) = \rho^\nu \overline{\boldsymbol{R}(\tau)}$, $\nu = 0, \ldots, 5$.

and

$$\boldsymbol{H}(\tau) = \frac{1}{2\pi i} \int_{|\sigma|=S} \frac{\boldsymbol{R}(\sigma)}{\sigma^6 - \tau^6} \left(\frac{\tau}{\sigma}\right)^{18} \tau\sigma^4 \, d\sigma \quad \text{when } S > |\tau|\,, \tag{17}$$

$$|\mathbf{H}(\tau)| \leq \frac{C \exp(DS)}{S^6 - |\tau|^6} \frac{|\tau|^{19}}{S^{13}} \quad \text{for all } S > |\tau|\,. \tag{18}$$

We write

$$\boldsymbol{R}(\tau) = \boldsymbol{F}(\tau) + \boldsymbol{H}(\tau), \quad \text{where } \boldsymbol{F}(\tau) = -2\,\tau - \frac{1}{12600}\tau^7 - \frac{17}{856215360000}\tau^{13} \tag{19}$$

and

$$\boldsymbol{H}(\tau) = \frac{1}{2\pi i} \int_{|\sigma|=S} \frac{\boldsymbol{R}(\sigma)}{\sigma^6 - \tau^6} \left(\frac{\tau}{\sigma}\right)^{18} \tau\sigma^4 \, d\sigma \quad \text{when } S > |\tau|\,. \tag{20}$$

The last expression comes from (13). By Lemma 4.3, we get that

$$|\boldsymbol{H}(\tau)| \leq \frac{C \exp(DS)}{S^6 - |\tau|^6} \frac{|\tau|^{19}}{S^{13}} \quad \text{for all } S > |\tau|\,. \tag{21}$$

where $C := \frac{3c}{\pi},\ D := \frac{I(-c)}{2}$.

Rouché theorem[14] leads to conclude that $\boldsymbol{R}$ and $\boldsymbol{F}$ have the same number of zeroes in the disk $|\tau| < 10.6$. Solving a quadratic equation, we easily find that $\boldsymbol{F}$ has exactly 7 zeroes in the disk and we have $\tau = 0$ and $\tau = \rho^\nu i\tau_1$, $\nu = 0, \ldots, 5$, where $\tau_1 \sim 5.4204$.

The first remark on the zeroes of $\boldsymbol{R}$ implies that they are located on the rays $\arg\tau = \frac{\pi}{6} + \nu\frac{\pi}{3},\ \nu = 0, \ldots, 5$. □

The zeroes of $\boldsymbol{R}$ satisfying $|\tau| > 10.6$ are less easily localized; we will use an asymptotic method: since $\boldsymbol{R}(\tau)$ is given by an integral formula (15) with an exponential term, we apply the "saddle method" to determine precise asymptotic estimates.

Lemma 4.6. *Consider the poles $B := -(1+\rho)c$ and $A := (-1+\rho^2)c$ of I and a path γ from A to B that goes through the saddle-point $-c$ of I, and along which* $\operatorname{Re}\{I(s)\}$ *is decreasing (and the imaginary part is constant). Let $\tilde{c} = I(-c)$. Then, for $|\arg\tau| < \frac{\pi}{2}$, we have*

$$\int_\gamma e^{\tau I(s)/2}\, ds = e^{\tilde{c}\tau/2}\big(-2i\sqrt{\pi}\,\tau^{-1/2} + \mathcal{R}(\tau)\big),$$

where

$$|\mathcal{R}(\tau)| \leq \frac{57}{20}\sqrt{\pi}\,(\operatorname{Re}\tau)^{-7/2}.$$

[14]Rouché theorem: If the two functions $f(z)$ and $\psi(z)$ are regular in a region G and if $|\psi(\xi)| < |f(\xi)|$ holds at every point ξ of the boundary γ of G, then the two functions $f(z)$ and $f(z) + \psi(z)$ have the same number of zeroes in G.

Proof: We use Laplace method with error bounds ([Olv], p 137) for large τ. $\square$

We apply Lemma 4.6 to the asymptotic estimation of $\boldsymbol{R}$: we deform the path γ and divide it in 4 pieces. We fix $0 \leq \arg\tau \leq \frac{\pi}{3}$. Using (15), we have

$$\boldsymbol{R}(\tau) = \frac{1}{2\pi i}\int_{\gamma+\gamma_1+\gamma_2+\gamma_3} e^{\tau I(s)/2}\,ds\ , \tag{22}$$

where γ is the path of Lemma 4.6, $\gamma_1 = \rho\gamma$ is the path from B to $C = -i\sqrt{3}\,c$ passing through $\rho^4 c$, $\gamma_2 = \gamma^{-1} + (1+\bar\rho)c$ is the path from C to 0 and $\gamma_3 = \gamma_1^{-1} + i\sqrt{3}\,c$ is the path from 0 to A. Since I is a pseudo-periodic function, namely $I(s+(1+\bar\rho)c) = I(s) - (1+\rho)\tilde c$ and $I(s+i\sqrt{3}\,c) = I(s) + i\sqrt{3}\,\tilde c$, and by symmetry, we prove, for $\rho\,\bar\rho^{-1/2} = i$, the following

Lemma 4.7. *For* $0 \leq \arg\tau \leq \frac{\pi}{3}$, *one has*

$$\begin{aligned}\boldsymbol{R}(\tau) &= \left(e^{\tilde c\tau/2} - e^{-\rho\tilde c\tau/2}\right)\left(-\frac{\tau^{-1/2}}{\sqrt{\pi}} + r(\tau)\right)\\ &\quad + \left(e^{\bar\rho\tilde c\tau/2} - e^{\rho\,\tilde c\tau/2}\right)\left(-i\frac{\tau^{-1/2}}{\sqrt{\pi}} + \rho r(\bar\rho\tau)\right)\end{aligned} \tag{23}$$

where $|r(\tau)| \leq \frac{57}{40\sqrt{\pi}}\,(\mathrm{Re}\,\tau)^{-7/2}$.

Now, we apply Rouché theorem once again. We write

$$-\sqrt{\pi}\tau^{1/2}\boldsymbol{R}(\tau) = \boldsymbol{F}(\tau) + \boldsymbol{H}(\tau), \tag{24}$$

where

$$\begin{aligned}&\boldsymbol{F}(\tau) = e^{\tilde c\tau/2} + i e^{\bar\rho\tilde c\tau/2},\\ &\boldsymbol{H}(\tau) = -e^{-\rho\tilde c\tau/2}\,(1 + r_1(\tau)) + e^{\tilde c\tau/2} r_1(\tau) - e^{\rho\,\tilde c\tau/2}\,(i + r_1(\bar\rho\tau)) + e^{\bar\rho\tilde c\tau/2} r_1(\bar\rho\tau).\end{aligned}$$

with $|r_1(\tau)| \leq \frac{57}{40}\,(\mathrm{Re}\,\tau)^{-7/2}\,|\tau|^{1/2}$. We prove that the zeroes of $\boldsymbol{R}$ are close to those of $\boldsymbol{F}$ given by $\tilde c\tau_k/2 = e^{\pi i/6}\left(\frac{3}{2}\pi + 2k\pi\right)$, $k \in \mathbb{Z}$. $\square$

Remark 4.1. We notice that the modulus of the first zeroes of $\boldsymbol{R}$ is almost equal to 5.4204. If we prove that the first zeroes of $\boldsymbol{R}$ really are the first singularities of $\boldsymbol{E}^*$, we will obtain (cf. section 2) that Δ^* is divergent, Gevrey of order 6 (in t^6) and the type is equal to $\frac{1}{5.4204} \simeq 0.1845$. We recall that the type $0.18\ldots$ was obtained numerically in Section 3.

4.4 The nonlinear problem

In order to find a solution of (6), it is sufficient to solve the fixed point equation $\boldsymbol{G} = \mathcal{G}(\mathcal{W}(\boldsymbol{G}), \mathcal{E}(\boldsymbol{G}))$ in $\tau^{-3}\,\mathcal{S}_6$, i.e. to solve

$$\begin{aligned}
&\boldsymbol{G} = \mathcal{F}(\boldsymbol{G}), \quad \text{where} \\
&\mathcal{F}(\boldsymbol{G}) = q(s)\tau * \mathcal{W}(\boldsymbol{G}) * \mathcal{W}(\boldsymbol{G}) - 2\mathcal{W}(\boldsymbol{G}) * \mathcal{E}(\boldsymbol{G}) - (\tau\mathcal{W}(\boldsymbol{G})) * \mathcal{E}(\boldsymbol{G}) \qquad (25) \\
&\qquad + \boldsymbol{F}_0 + \sum_{n=1}^{\infty} \boldsymbol{F}_n * \tau^{n-1} * \mathcal{W}(\boldsymbol{G})^{*n}/(n-1)!.
\end{aligned}$$

Let $\mathcal{D} = \mathbb{C} \setminus \bigcup_{\mu=0}^{5} e^{\frac{\pi}{6}i}\, e^{\mu\frac{\pi}{3}i}\, [m_0, \infty[$.

Theorem 4.8. *There exists a unique analytic solution* $\boldsymbol{G} : (\mathbb{C} \setminus \mathcal{R}) \times \mathcal{D} \to \mathbb{C}$ *of (25). It is* $\mathcal{R}$*-periodic*[15]*, symmetric:* $\boldsymbol{G}(\rho s, \rho\tau) = \rho^{-3}\boldsymbol{G}(s,\tau) = -\boldsymbol{G}(s,\tau)$ *and* $\boldsymbol{G}(s,\tau) \in \tau^{-3}\,\mathcal{S}_6$.

As a consequence $\boldsymbol{G}$, thus also $\mathcal{W}(\boldsymbol{G}) = \boldsymbol{W}$, $\mathcal{E}(\boldsymbol{G}) = \boldsymbol{E}^*$ have convergent power series expansions at $\tau = 0$, hence by Section 2, Proposition 2.4, the formal solution (Δ^*, U) of (4) is Gevrey 1. We prove the theorem by showing that $\mathcal{F}$ defines a contraction on some subset of the following Banach space: For small $\delta > 0$ and large $M > 0$, let $\mathcal{D}_\delta^M$ denote the set of all τ, $|\tau| < M$, such that τ does not lie in any sector $|\arg\tau - j\frac{\pi}{3} - \frac{\pi}{6}| \le \delta$, $j = 0, \ldots, 5$ if $|\tau| \ge m_0 - \delta$ and let $\mathcal{C}_\delta$ be the set of all $s \in \mathbb{C}$ such that $\operatorname{dist}(s, \mathcal{R}) > \delta$. Then $\mathcal{O} = \{V : \mathcal{D}_\delta^M \times \mathcal{C}_\delta \to \mathbb{C} \mid V$ holomorphic, bounded, $\mathcal{R}$-periodic, $\tau^{-3}V$ holomorphic at $\tau = 0\}$ is a Banach space endowed with the family of equivalent norms

$$||V||_L = \sup_{s\in\mathcal{C}_\delta,\ \tau\in\mathcal{D}_\delta^M} |V(s,\tau)|\, e^{-L|\tau|}, \quad L > 0.$$

We first show that $\mathcal{W}$ and $\mathcal{E}$ are bounded linear operators on $\mathcal{O}$, the norms of which do not depend on L. Then we show that $\mathcal{F}$ defines a contraction on some neighbourhood of 0 in $\mathcal{O}$ for sufficiently large L. Since δ and M are arbitrary, the theorem follows.

Theorem 4.9. *Let* $\theta \in \mathbb{R} \setminus (\frac{\pi}{6} + \frac{\pi}{3}\mathbb{Z})$*. For sufficiently small* $\delta > 0$*, there exists* $K, M > 0$ *such that* $|\boldsymbol{G}(s,\tau)| \le M\exp(K|\tau|)$ *for all* $s, \tau \in \mathbb{C}$ *with dist*$(s, \mathcal{R}) > \delta$ *and* $|\arg\tau - \theta| < \delta$.

Note that this theorem implies that also $\boldsymbol{E}^* = \mathcal{E}(\boldsymbol{G})$ has at most exponential growth as $\tau \to \infty$. This proves the 1-summability of $A_0(t^6)$ stated in Theorem 4.2. We also obtain the 1-summability of $\tilde{U}(s,t) = (\mathcal{C}U)(s,t)$ with respect to t, but it would be too cumbersome here to state this in terms of x, y.

The idea of the proof of Theorem 4.9 is to introduce a certain subset $\mathcal{A}_\theta \subset \mathbb{C}$ of s that can be joined to ∞ by a path $\gamma(\sigma)$, $-\infty < \sigma \le 0$ on which $\operatorname{Re}(e^{\theta i} I(\sigma))$ increases. Then we obtain an inequality containing convolutions for

[15] $\mathcal{R} = \mathbb{Z}p_0 + \mathbb{Z}p_1$ is the lattice of periods of q.

$f(r) = \sup\{|\boldsymbol{G}(s,t)| \mid s \in \mathcal{A}_\theta, |\arg\tau - \theta| < \delta, |\tau| \le r\}$. Following an idea of B. L. J. Braaksma and W. Walter, we show, using the Laplace transform, that the corresponding equation has a solution with at most exponential growth and that this solution is a majorant of $f(r)$. Finally we discuss the case $s \in \mathbb{C} \setminus \mathcal{A}_\theta$.

4.5 Divergence

In order to show the divergence of the formal solution in the generic case, it is sufficient to show that its Borel transform $\boldsymbol{E}^*(\tau)$ has a singularity at $\tau = \tau_j = m_0 \exp(\frac{\pi}{6}i + j\frac{\pi}{3}i)$, $j = 0, \dots, 5$. This case is done in a way similar to the preceding proof, but working with a different function space.

Therefore we consider the subset $\tilde{D}$ of the universal covering $\hat{D}$ of $\{\tau \in \mathbb{C} \mid |\tau| < m_0 + 1\} \setminus \{\tau_0, \dots, \tau_5\}$ consisting of all points that can either be joined (in $\hat{D}$) to 0 by a segment or whose distance to one of the τ_j, $j = 0, \dots, 5$ is smaller than 1. We write this decomposition $\tilde{D} = \check{D} \cup \bigcup_{j=0}^5 B_j$. Let $\mathcal{O}$ the Banach space of all holomorphic functions $\boldsymbol{G} : \mathcal{C}_\delta \times \tilde{D} \to \mathbb{C}$ bounded on $\mathcal{C}_\delta \times \check{D}$ that can be written in the form

$$\boldsymbol{G}(s,\tau) = \alpha_j^{\boldsymbol{G}}(s,\tau) + \beta_j^{\boldsymbol{G}}(s,\tau)(\tau-\tau_j)\log(\tau-\tau_j), \;\; 0 < |\tau-\tau_j| < 1, \;\; j = 0, \dots, 5,$$

where $\alpha_j^{\boldsymbol{G}}$ and $\beta_j^{\boldsymbol{G}}$ are bounded holomorphic functions on $\mathcal{C}_\delta \times \{|\tau - \tau_j| < 1\}$.

We show that (25) has a unique solution $\boldsymbol{G}$ in the closed subspace $\mathcal{O}_{6,6}$ of all $\boldsymbol{H} \in \mathcal{O}$ such that $\tau^{-6}\boldsymbol{H}(s,\tau)$ and all $(\tau - \tau_j)^{-6}\beta_j^{\boldsymbol{H}}(s,\tau)$ are bounded. By (9), there exists a constant $a_0 \in \mathbb{C}$ such that $\boldsymbol{E}^* - \sum_{j=0}^5 \frac{a_0\rho^{-2j}}{\tau-\tau_j} \in \mathcal{O}_{6,5}$. We show that a_0 depends analytically on the coefficients of Δ and, by considering the example $\Delta = x + \varepsilon y$, $\varepsilon \neq 0$ small, we show that a_0 is a non trivial function. By Section 2, Proposition 2.4, this proves the divergence of E^* (hence of Δ^* and consequently U) in the generic case and yields the function $\mathcal{Q}$ of the theorem.

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Application of J.-J. Morales and J.-P. Ramis' theorem to test the non-complete integrability of the planar three-body problem

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Abstract. We give a proof of the non complete integrability of the planar three-body problem. We use a criterion of non complete integrability deduced from J.-J. Morales and J.-P. Ramis' theorem and based on a parameterized linear differential system called variational system. We show how to apply this criterion to the three-body problem using symbolic computation's tools for parameterized linear differential systems. Our strategy consists in studying locally and globally this normal variational system without transforming it into an equivalent linear differential equation.

1 Introduction

Let us consider three bodies in a newtonian reference system and let us assume that the only forces acting on them are their mutual gravitational attraction. Each body is represented by its mass m_i, its position q_i and its moment p_i ($p_i = m_i \frac{dq_i}{dt}$). According to the Newton law and the law of gravitation the equations of the motion of these bodies can be written in the following form:

$$\left\{ \begin{array}{l} \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \\ \\ \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \end{array} \right. \qquad i = 1, 2, 3$$

where

$$H = \sum_{j=1}^{3} \frac{\|p_j\|^2}{2m_j} - \sum_{1\leq j<k\leq 3} \frac{m_j m_k}{\|q_j - q_k\|}.$$

These equations form a differential system, called *Hamiltonian system*. It depends on the three parameters m_1, m_2 and m_3. In the sequel, we choose a normalization by assuming (without loss of generality) that $m_3 = 1$. The *Hamiltonian H* represents the mechanical energy and is conserved. One says that H is a *first integral* for the Hamiltonian system. If one can find a sufficient number of first integrals satisfying independence and involution properties then one may assume that the Hamiltonian system will have a non-chaotic behavior in the studied region and one says that it will be *completely integrable* ([M-R], [Au1]). If we assume that the three bodies are in a plane, we can reduce the number of variables of the initial Hamiltonian ([Tsy1]). We also assume that the constant of the cinetic moment is non zero and without loss of generality we fix it equal to one. We then work with the following Hamiltonian (that we denote H again):

$$H = \tfrac{1}{2}\left(\tfrac{1}{m_1}+1\right)\left(p_1{}^2 + \tfrac{(p_3q_2-p_2q_3-1)^2}{q_1{}^2}\right) + \tfrac{1}{2}\left(\tfrac{1}{m_2}+1\right)(p_2{}^2+p_3{}^2) + p_1p_2$$
$$- \tfrac{p_3(p_3q_2-p_2q_3-1)}{q_1} - \tfrac{m_2}{\sqrt{q_2{}^2+q_3{}^2}} - \tfrac{m_1}{q_1} - \tfrac{m_1m_2}{\sqrt{(q_1-q_2)^2+q_3{}^2}}.$$

The non-integrability of this Hamiltonian system will imply the non-integrability of the planar three-body problem.

In 1890, Poincaré proved that the three-body problem does not have any additional analytic first integral besides the known integrals ([Poin]). To obtain this result, he studied a *variational system* which is a *linear* differential system computed along a particular solution of the Hamiltonian system.

Definition 1. The *variational system* along a solution $\mathrm{x}_0(t)$ of a Hamiltonian system is the linear differential system:

$$Y'(t) = A(t)Y(t)$$

where

$$A(t) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \mathcal{H}(H, \mathrm{x}_0(t)).$$

$\mathcal{H}(H, \mathrm{x}_0(t))$ is the Hessian of H at $\mathrm{x}_0(t)$.

We emphasize on the special structure of this variational system (the matrix A is infinitesimally symplectic), which makes further computations much easier despite the presence of parameters in the matrix A.

During the last twenty years many significant improvements regarding complete (meromorphic) integrability of Hamiltonian systems have been obtained by Ziglin

([Zig1], [Zig2]) in 1982, by Baider, Churchill, Rod and Singer ([C-R-S]) in 1996, and by Morales and Ramis ([M-R]) in 1998. They all found necessary conditions of complete (meromorphic) integrability based on the monodromy group ([Zig1], [Zig2]) or the differential Galois group ([C-R-S], [M-R]) of the *variational system*. Our study will rely on the theorem of Morales and Ramis ([M-R], [Au1]) :

Theorem 1. *Let* (S) *be a Hamiltonian system,* $\mathrm{x}_0(t)$ *a particular solution of* (S), $Y'(t) = A(t)Y(t)$ *the variational system of* (S) *computed along the solution* $\mathrm{x}_0(t)$ *and* G *the differential Galois group of* $Y'(t) = A(t)Y(t)$.

If the system (S) *is completely integrable with meromorphic first integrals, then the connected component of the identity in the group* G*, denoted* G^0*, is an abelian group.*

This theorem remains true if one replaces the *variational system* with the *normal variational system*. It is obtained after a standard symplectic transformation which reduces the order of the variational system but keeps its infinitesimally symplectic structure ([Au1]).

We deduced from this theorem a criterion based on a local and global formal study (detection of logarithms and factorization) of the *normal variational system* ([Bou2], [BW02]).

Criterion 1. *Let* (S) *be a Hamiltonian system and let* $Y'(t) = A(t)Y(t)$ *be the normal variational system computed along a particular solution of* (S).

If the normal variational system $Y'(t) = A(t)Y(t)$ *has a completely reducible factor whose local solutions at a singular point contain logarithmic terms* (*or exhibit a non-trivial Stokes phenomenon*), *then the Hamiltonian system* (S) *is not completely integrable* (*with meromorphic first integrals*).

In the next section we will apply this criterion to the planar three-body problem. Instead of transforming the normal variational system into a linear differential equation ([Bou1], [Bou2], [BW02]), we keep the structure of the system and make direct computations on it. This gives a new proof on non-integrability of the three-body problem, which we present to emphasize on the method used, which we believe should be fruitful on other problems (such studies are currently in progress).

Previous proofs of non-integrability were given by Alexei Tsygvints'ev in [Tsy1], [Tsy2], [Tsy3] and in parallel by the authors in [Bou1] and [BW02]. We enjoy noting that these parallel proofs were obtained thanks to active friendly and open discussions with Tsygvints'ev: collaborating nicely led to two nice proofs. We thank Alexei for this, and Jean-Pierre Ramis for suggesting this beautiful problem to us.

2 Application of the criterion to the planar three-body problem

2.1 The normal variational system

We take the Lagrange solution as particular solution of the Hamiltonian system associated to H ([Tsy1]). The three bodies then form a configuration which is homographically equivalent to an equilateral triangle: each body describes a parabola centered on a vertex of the equilateral triangle.

After reductions, the normal variational system is a 4×4 linear differential system

$$Y'(x) = A(x)Y(x)$$

(see Annex 1).

2.2 Factorization of the normal variational system

Let us first define factorization. The system $Y'(x) = A(x)Y(x)$ is called *equivalent* to a system $Z'(x) = B(x)Z(x)$ if one is obtained from the other by a gauge transformation, i.e. a change of variable $Z = PY$ with P an invertible square matrix with rational coefficients. The system is called *factorizable* (or reducible) if it is equivalent with a block triangular system (and *irreducible* otherwise), and it is called *decomposable* if it is equivalent with a block diagonal system with smaller blocks; it is called *completely reducible* if it is equivalent with a block diagonal system where blocks are irreducible (note that this includes the irreducible case).

To find factors of size 2×2 of our 4×4 normal variational system we use the second exterior system (see Annex B2 of [CW] for example or [PS02]) and we give some links to the factorization using the eigenring ([Bar2], [Pflu]).

- If $(m_1, m_2) = (1, 1)$ we get a linear differential system without parameter. It is equivalent to the system

  $$Z' = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix} Z$$

  where the blocks F_1 and F_2 are given in Annex 2A.

  Each block F_1 and F_2 is irreducible so the system is completely reducible.

  We do not explain the computations here and choose to give them in the following second case.

- If $(m_1, m_2) \neq (1, 1)$, then we get two distinct situations that we detail below. The second exterior system has two exponential (in fact, rational) solutions $W_I = (0, 1, 0, 0, 1, 0)$ and W whose expression is too big to be given here. The vector $W - \lambda W_I$ induces a 2×2 factor for the system $Y' = AY$ for values of λ such that $W - \lambda W_I$ is a "pure tensor" for the second exterior system. Following

Annex B2 of [CW], we see that these values of λ are the ones which annihilate the determinant (the associated "plucker relation") of the matrix

$$M_\lambda = \begin{pmatrix} W[4] & -(W[2]-\lambda) & W[1] & 0 \\ W[5]-\lambda & -W[3] & 0 & W[1] \\ W[6] & 0 & -W[3] & W[2]-\lambda \\ 0 & W[6] & -(W[5]-\lambda) & W[4] \end{pmatrix}.$$

This gives use two values for λ :

$$\lambda_1 = \frac{(m_1+1-2m_2+2r)r(2m_1^2+2m_2^2+2-5m_1-5m_2-5m_1m_2)}{2(m_1^2+m_2^2+1-m_1-m_2-m_1m_2)}$$

and

$$\lambda_2 = \frac{(m_1+1-2m_2-2r)r(2m_1^2+2m_2^2+2-5m_1-5m_2-5m_1m_2)}{2(m_1^2+m_2^2+1-m_1-m_2-m_1m_2)}$$

where r satisfies

$$r^2 = m_1^2+m_2^2-m_1m_2-m_1-m_2+1. \tag{1}$$

Remark 1. The vector W_I corresponds to the identity matrix, and the vector W corresponds to a matrix T in the eigenring $\mathcal{E}(A)$ of A (see [Sin1], [Bar2], [Pflu]). The values λ_1 and λ_2 are also the eigenvalues of the matrix T. One can even notice

$$T-\lambda I = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} M_\lambda.$$

There are now two cases,whether the λ_i are distinct or not:

Decomposable case. If $2m_1^2+2m_2^2+2-5m_1-5m_2-5m_1m_2 \neq 0$, then λ_1 and λ_2 are distinct and we have two pure tensors $W-\lambda_1 W_I$ and $W-\lambda_2 W_I$ *(i.e. two matrices of rank* 2 *in* $\mathcal{E}(A)$*,* $T-\lambda_1 I$ *and* $T-\lambda_2 I$*).* We construct the matrix P with two vectors generating the kernel of M_{λ_1} and two vectors generating the kernel of M_{λ_2}. After the gauge transformation $Y = PZ$ we get the following equivalent linear differential system:

$$Z' = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix} Z.$$

The coefficients of F_1 are given in Annex 2B, and the factor F_2 is obtained from the factor F_1 after replacing r with $-r$.

Let us now prove that the factors F_1 and F_2 cannot be simultaneously reducible.

We first focus on the block F_1. It is irreducible if and only if it has no exponential solution. The singular points of the system are ∞, i, $-i$ and 0.

It turns out that our system is locally fuchsian (at each point). So, after Moser reduction (see [Mos], [Bar1]), we can reduce this system to one with simple poles.

For such a system with a simple pole p (i.e of the form $Y'(x) = \frac{B(x)}{x-p}\,Y(x)$ with B analytic at p), the *indicial equation* at p is the characteristic polynomial of $B(p)$ and the *exponents at p* are the eigenvalues of $B(p)$. This will always be the case in the sequel.

If $m_1 \neq 1$, the indicial equation at infinity is

$$(m_1+m_2+1)(m_1^2+m_2^2-m_1m_2-m_1-m_2+1)(m_1-1)^2m_2m_1^2$$
$$((m_1+m_2+1)(\alpha^2-3\alpha-1)+3r)=0.$$

If $m_1 = 1$ ($r = m_2 - 1$), the indicial equation at infinity is:

$$(m_2+2)(\alpha^2-3\alpha-1)+3m_2-3=0.$$

So the indicial equation at infinity for the factor F_1 is:

$$(m_1+m_2+1)(\alpha^2-3\alpha-1)+3r=0.$$

The exponents at i and $-i$ are -2 and -1.

The exponents at 0 are -1 and 0.

As the exponents at i, $-i$ and 0 are integers, the factor F_1 (resp. F_2) is reducible only if the indicial equation at infinity has an integer solution n_1 (resp. n_2), see [Pflu], [Bou2].

But the equations

$$\begin{cases} (m_1+m_2+1)(n_1^2-3n_1-1)+3r=0 & \text{(equation for } F_1\text{)} \\ (m_1+m_2+1)(n_2^2-3n_2-1)-3r=0 & \text{(equation for } F_2\text{)} \end{cases}$$

imply

$$(m_1+m_2+1)(n_1^2+n_2^2-3n_1-3n_2-2)=0,$$

and as $m_1+m_2+1 \neq 0$,

$$(2n_1-3)^2+(2n_2-3)^2=26.$$

The solutions of this equation in $\mathbb{Z}\times\mathbb{Z}$ are $(2,4)$, $(2,-1)$, $(1,4)$, $(1,-1)$, $(4,2)$, $(-1,2)$, $(4,1)$ and $(-1,1)$.

But

$$n_1 \in \{2,1\} \Rightarrow m_1+m_2+1-r=0$$
$$n_1 \in \{4,-1\} \Rightarrow m_1+m_2+1-r=0$$

so

$$n_1 \in \{4,2,1,-1\} \Rightarrow r^2=(m_1+m_2+1)^2.$$

According to (1), one gets $m_1+m_2+m_1m_2=0$, which is excluded as m_1 and m_2 are positive (recall that they represent masses in the 3-body problem, so they must be *positive real number*).

To conclude, the factors F_1 and F_2 cannot be simultaneously reducible.

Factorizable case. If $2m_1^2 + 2m_2^2 + 2 - 5m_1 - 5m_2 - 5m_1m_2 = 0$, we introduce a new parameter s such that:

$$m_1 = \frac{(5+3\sqrt{3})(s-2+\sqrt{3})(s-7+4\sqrt{3})}{(1-s)(1+s)}$$

$$m_2 = \frac{(5+3\sqrt{3})(s+2-\sqrt{3})(s+7-4\sqrt{3})}{(1-s)(1+s)}$$

with

$$2-\sqrt{3} < |s| < 1 \quad \text{or} \quad |s| < 7-4\sqrt{3}.$$

The vector W is again solution of the second exterior system and it is a pure tensor $(W - \lambda_1 W_I = W - \lambda_2 W_I = W)$. The kernel of the matrix $M_{\lambda_1} = M_{\lambda_2} = M_0$ is generated by two vectors. One completes these two vectors into a basis to get the gauge transformation matrix P.

Remark 2. One finds an element T in the eigenring $\mathcal{E}(A)$ of A associated to the solution W and it has one single eigenvalue, 0.

One gets the equivalent linear differential system:

$$Z' = \begin{pmatrix} F & * \\ 0 & * \end{pmatrix} Z.$$

The coefficients of F are given in Annex 2C.

The exponents of the singular points ∞, $-i$, i and 0 are :

at infinity : the roots of $\alpha^2 - 2\alpha - 1 = 0$,

at 0: -1 and 1,

at i and $-i$: -2 and -1.

As the equation $\alpha^2 - 2\alpha - 1 = 0$ has no integer solution, the system $Y' = FY$ has no exponential solution and the factor F is irreducible.

2.3 Formal solutions with logarithmic terms

We prove that each factor F_1, F_2 and F has formal solutions with logarithmic terms at the point i.

We study each of the previous cases.

- $(m_1, m_2) = (1, 1)$
  We get two formal solutions with logarithmic terms which are linearly independant (computations made using the package Isolde in the computer algebra system `maple`, see [Pflu]).

- $(m_1, m_2) \neq (1, 1)$
  - $2m_1^2 + 2m_2^2 + 2 - 5m_1 - 5m_2 - 5m_1m_2 \neq 0$
    Let us prove that the factors F_1 and F_2 both have one formal solution at the point i with logarithmic terms. The factor F_1 has two exponents $\rho_0 = -1$ and $\rho_1 = 1$ at the point i. As they differ from an integer there may be formal solutions with logarithmic terms. After moving the singularity i to the point 0, one can compute an equivalent 2×2 linear system:

    $$xY'(x) = (N_0 + N_1x + \cdots + N_kx^k + \cdots)Y(x)$$

    We want to find the number of formal solutions $Y(x) = x^{\rho_0}(Y_0 + Y_1x + \cdots)$ which are linearly independent and without logarithmic term. The coefficients Y_k satisfy the following recurrence relation:

    $$((k + \rho_0)I - N_0)Y_k = \sum_{j=1}^{k} N_jY_{k-j}, \quad k \in \mathbb{N}.$$

    The coefficient Y_k is uniquely determined when $k + \rho_0 > \rho_1$. The $\rho_1 - \rho_0 + 1 = 3$ first equations can be written

    $$\mathcal{M}\mathcal{Y} = 0$$

    where $\mathcal{Y}$ is the vector ${}^t(Y_0, \ldots, Y_{\rho_1-\rho_0})$ and where $\mathcal{M}$ is the following 6×6 matrix:

    $$\mathcal{M} = \begin{pmatrix} -I - N_0 & 0 & 0 \\ -N_1 & -N_0 & 0 \\ -N_2 & -N_1 & I - N_0 \end{pmatrix}.$$

    Its determinant is zero and the 5×5 matrix obtained from the rows $2, 3, 4, 5, 6$ and the columns $1, 2, 3, 4, 5$ has the following determinant:

    $$1152i(m_1 + m_2 + m_1m_2)(m_2^2 - m_1m_2 - m_2 - m_1 + m_1^2 + 1)^5$$

    It is non zero for each (m_1, m_2) in $\mathbb{R}_+^* \times \mathbb{R}_+^* - \{(1, 1)\}$.
    The kernel of the matrix $\mathcal{M}$ has one single element and the 2×2 system $Y' = F_1Y$ (resp. $Y' = F_2Y$) has $2 - 1$ formal solution with a logarithmic term.
  - $2m_1^2 + 2m_2^2 + 2 - 5m_1 - 5m_2 - 5m_1m_2 = 0$
    We study the factor F. We again have two exponents $\rho_0 = -1$ and $\rho_1 = 1$ at the point i. We construct the matrix $\mathcal{M}$ from which we extract the matrix with the rows $2, 3, 4, 5, 6$ and the columns $1, 2, 3, 4, 5$. The determinant of this submatrix is

    $$(3 + i\sqrt{3})(s + i\sqrt{3} - 2i)4(s + 2i - i\sqrt{3})$$

    and it is non zero. So again there is a formal solution with logarithmic terms for the 2×2 system $Y' = FY$.

2.4 Conclusion

We recall that both the parameters m_1 and m_2 are positive real. We have three situations:

- If $(m_1, m_2) = (1, 1)$ then the normal variational system has no parameter, it is completely reducible and has formal solutions with logarithmic terms at the point i.

- If $(m_1, m_2) \neq (1, 1)$ and $2m_1^2 + 2m_2^2 + 2 - 5m_1 - 5m_2 - 5m_1m_2 \neq 0$ then the normal variational system is *decomposable* i.e. equivalent to a block diagonal 4×4 system of the form

$$Z' = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix} Z.$$

  The factors F_1 and F_2 both depend on the parameters m_1 and m_2. They cannot be simultaneously reducible and they both have formal solutions with logarithmic terms at the point i.

- If $(m_1, m_2) \neq (1, 1)$ and $2m_1^2 + 2m_2^2 + 2 - 5m_1 - 5m_2 - 5m_1m_2 = 0$ then the normal variational system is *factorizable* i.e. equivalent to a system

$$Z' = \begin{pmatrix} F & * \\ 0 & * \end{pmatrix} Z.$$

  The factor F depends on the parameters, it is irreducible and has formal solutions with logarithmic terms at the point i.

We thus obtain the following theorem, which derives immediately from Criterion 1. and the above results :

Theorem 2. *The planar three-body problem is not meromorphically completely integrable. known ones.*

The tools we have used overall were: quite big symbolic computations (but relatively easy for a computer) directed from mathematical insight of the problem, a small but crucial physical hypothesis (masses are positive real numbers), and relatively simple mathematical criteria. It turns out that many families of systems provided by Newtonian mechanics exhibit the features that we have heavily used (the variational equation is reducible, is relatively easy to factor for a computer directed by an expert hand, and there are logarithms in formal solutions). We thus believe that other expert hands than ours might fruitfully use this scheme of thought to prove other non-integrability results.

Annex 1

Normal variational system along Lagrange' solution

$$Y'(x) = \begin{pmatrix} s_1 & s_2 & s_5 & 0 \\ s_3 & s_4 & 0 & s_5 \\ & & & \\ s_6 & s_7 & -s_1 & -s_3 \\ s_7 & s_8 & -s_2 & -s_4 \end{pmatrix} Y(x)$$

with

- $s_1 = \frac{m_2\left(\left(-5{m_1}^2-5-14m_1\right)x^2+4\sqrt{3}(m_1-1)(m_1+1)x-(m_1-1)^2\right)}{4(m_1+m_2+1)m_1x(x^2+1)}$

- $s_2 = \frac{-3\sqrt{3}(m_1-1)(m_1+1)m_2x^2-4(m_1+1)(m_1m_2-2m_1+m_2)x+\sqrt{3}(m_1-1)(m_1+1)m_2}{4(m_1+m_2+1)m_1x(x^2+1)}$

- $s_3 = \frac{-3\sqrt{3}(m_1-1)(m_1+1)m_2x^2+\left(-4{m_1}^2m_2-8m_1-4m_2-24m_1m_2-8{m_1}^2\right)x+\sqrt{3}(m_1-1)(m_1+1)m_2}{4(m_1+m_2+1)m_1x(x^2+1)}$

- $s_4 = \frac{m_2\left(\left({m_1}^2+1+10m_1\right)x^2-4\sqrt{3}(m_1-1)(m_1+1)x-3(m_1+1)^2\right)}{4(m_1+m_2+1)m_1x(x^2+1)}$

- $s_5 = \frac{(m_1+m_2+1)(x^2+1)}{2m_1m_2(m_1+m_2+m_1m_2)2}$

- $s_6 = \Delta\,(m_1+1)\,\frac{\left(-13{m_1}^2m_2-2{m_1}^2-24m_1m_2-2m_1-13m_2\right)x^2+4\sqrt{3}(m_1-1)(m_1+1)m_2x-m_2(m_1-1)^2}{\left(1+x^2\right)^2x^2}$

- $s_7 = \Delta\,\frac{\left(-3\sqrt{3}\left({m_1}^2m_2+2{m_1}^2+4m_1m_2+2m_1+m_2\right)(m_1-1)x^2\right.}{\left(1+x^2\right)^2x^2}$
$$\frac{\left.-4m_2(m_1+1)\left({m_1}^2+4m_1+1\right)x+\sqrt{3}(m_1-1)m_2(m_1+1)^2\right)}{\left(1+x^2\right)^2x^2}$$

- $s_8 = \Delta(m_1+1)\,\frac{\left(-7{m_1}^2m_2+10{m_1}^2+12m_1m_2+10m_1-7m_2\right)x^2-4\sqrt{3}(m_1-1)(m_1+1)m_2x-3m_2(m_1+1)^2}{\left(1+x^2\right)^2x^2}$

where $\Delta = \frac{m_2(m_1m_2+m_2+m_1)^3}{2m_1(m_1+m_2+1)^3}$.

Annex 2

Factorization of the normal variational system

- Annex 2A

$$\boxed{\text{Case } m_1 = m_2 = 1}$$

$$Z' = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix} Z$$

with

$$F_1 = \begin{pmatrix} -\frac{(3x^2+2x+1)(2x^4-5x^2+6x-1)}{x(3x^4-6x^2-1)(x^2+1)} & -\frac{(x^2-2x-1)(3x^4-2x^3+3x^2+2)}{x(3x^4-6x^2-1)(x^2+1)} \\ -\frac{x(9+20x^3+12x+9x^4+18x^2)}{(3x^4-6x^2-1)(x^2+1)} & -\frac{-4+3x^5-26x^3-5x-4x^4-8x^2}{(3x^4-6x^2-1)(x^2+1)} \end{pmatrix}$$

$$F_2 = \begin{pmatrix} -\frac{(3x^2-2x+1)(2x^4-5x^2-6x-1)}{x(3x^4-6x^2-1)(x^2+1)} & \frac{(-1+2x+x^2)(3x^4+2x^3+3x^2+2)}{x(3x^4-6x^2-1)(x^2+1)} \\ \frac{x(9-20x^3-12x+9x^4+18x^2)}{(3x^4-6x^2-1)(x^2+1)} & -\frac{4+3x^5-26x^3-5x+4x^4+8x^2}{(3x^4-6x^2-1)(x^2+1)} \end{pmatrix}$$

- Annex 2B

$$\boxed{\begin{array}{c} \text{Case } (m_1, m_2) \neq (1, 1) \text{ and} \\ 2m_1^2 + 2m_2^2 + 2 - 5m_1 - 5m_2 - 5m_1m_2 \neq 0 \end{array}}$$

$$r^2 = m_1^2 + m_2^2 - m_1m_2 - m_1 - m_2 + 1$$

$$Z' = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix} Z$$

$$F_2(m_1, m_2, r) = F_1(m_1, m_2, -r)$$

and the entries of the matrix F_1 are given by

$$\begin{aligned} \Delta\mathbf{F_1}[\mathbf{1},\mathbf{1}] = &-r\sqrt{3}(2 - 5m_2 + 2{m_2}^2 - 5m_1 - 5m_1m_2 + 2{m_1}^2)(m_1 - 1)x^4 \\ &- 12r(m_1 + m_2 + 1)(m_1m_2 + m_2 - {m_1}^2 - 1)x^3 \\ &- 6r\sqrt{3}(2{m_1}^2 + m_1 + 2 + 2{m_2}^2 + m_1m_2 + m_2)(m_1 - 1)x^2 \\ &+ 4r(m_1 + m_2 + 1)(4{m_2}^2 - m_2 - m_1m_2 - 4m_1 + {m_1}^2 + 1)x \\ &- r\sqrt{3}(2 - 5m_2 + 2{m_2}^2 - 5m_1 - 5m_1m_2 + 2{m_1}^2)(m_1 - 1) \end{aligned}$$

$$
\begin{aligned}
\Delta\mathbf{F_1}[\mathbf{1},\mathbf{2}] = &-(2-5m_2+2{m_2}^2-5m_1-5m_1m_2+2{m_1}^2)(-2{m_2}^2+2rm_2+2m_2\\
&+2m_1m_2-2{m_1}^2-r-m_1r+2m_1-2)x^4\\
&+4\sqrt{3}(m_1-1)r(m_1+m_2+1)^2x^3\\
&+(12m_1m_2+28{m_1}^3r-28{m_2}^3+8-18rm_2+18m_1r-28{m_1}^3m_2-28{m_1}^3\\
&+12{m_2}^2m_1-28m_1-18{m_1}^2rm_2+36{m_2}^2+36{m_1}^2+36{m_2}^2{m_1}^2+18{m_1}^2r\\
&-48m_1rm_2+28r+8{m_2}^4+8{m_1}^4-28{m_2}^3m_1-28m_2+12{m_1}^2m_2-8r{m_2}^3)x^2\\
&-4\sqrt{3}(m_1-1)r(m_1+m_2+1)^2x\\
&-48m_1rm_2+12m_1r{m_2}^2-9{m_1}^2rm_2+6m_1m_2+4-14{m_2}^3m_1-3{m_1}^2r+18{m_2}^2{m_1}^2\\
&+6{m_2}^2m_1-3m_1r+18{m_1}^3r+12r{m_2}^2+12r{m_2}^3-9rm_2+6{m_1}^2m_2-14{m_1}^3m_2+18r\\
&-14m_2-14m_1+18{m_2}^2+18{m_1}^2-14{m_1}^3-14{m_2}^3+4{m_2}^4+4{m_1}^4
\end{aligned}
$$

$$
\begin{aligned}
\Delta\mathbf{F_1}[\mathbf{2},\mathbf{1}] = &-(2-5m_2+2{m_2}^2-5m_1-5m_1m_2+2{m_1}^2)(2{m_2}^2+2rm_2-2m_2-2m_1m_2+2{m_1}^2\\
&-r-m_1r-2m_1+2)x^4\\
&+4\sqrt{3}(m_1-1)r(m_1+m_2+1)^2x^3\\
&+(-12m_1m_2-4{m_1}^3r+28{m_2}^3-8-18rm_2+18m_1r+28{m_1}^3m_2+28{m_1}^3-12{m_2}^2m_1+28m_1\\
&-18{m_1}^2rm_2-36{m_2}^2-36{m_1}^2-36{m_2}^2{m_1}^2+18{m_1}^2r+48m_1rm_2-4r-8{m_2}^4-8{m_1}^4\\
&+28{m_2}^3m_1+28m_2-12{m_1}^2m_2-40r{m_2}^3)x^2\\
&-4\sqrt{3}(m_1-1)r(m_1+m_2+1)^2x\\
&-4+48m_1rm_2+12m_1r{m_2}^2-9{m_1}^2rm_2+14{m_2}^3m_1-3{m_1}^2r-18{m_2}^2{m_1}^2-6m_1m_2-3m_1r\\
&-14{m_1}^3r+12r{m_2}^2-6{m_2}^2m_1-9rm_2-6{m_1}^2m_2+14{m_1}^3m_2-20r{m_2}^3-14r+14m_2+14m_1\\
&-18{m_2}^2-18{m_1}^2+14{m_1}^3+14{m_2}^3-4{m_2}^4-4{m_1}^4
\end{aligned}
$$

$$
\begin{aligned}
\Delta\mathbf{F_1}[\mathbf{2},\mathbf{2}] = &\, r\sqrt{3}(2-5m_2+2{m_2}^2-5m_1-5m_1m_2+2{m_1}^2)(m_1-1)x^4\\
&+4r(m_1+m_2+1)(4{m_2}^2-m_2-m_1m_2-4m_1+{m_1}^2+1)x^3\\
&+6r\sqrt{3}(2{m_1}^2+m_1+2+2{m_2}^2+m_1m_2+m_2)(m_1-1)x^2\\
&-12r(m_1+m_2+1)(m_1m_2+m_2-{m_1}^2-1)x\\
&+r\sqrt{3}(2-5m_2+2{m_2}^2-5m_1-5m_1m_2+2{m_1}^2)(m_1-1)
\end{aligned}
$$

where $\Delta = 8(x^2+1)^2 r(m_2^2 - m_1m_2 - m_2 - m_1 + m_1^2 + 1)(m_1+m_2+1)$.

- Annex 2C

$$
\boxed{\begin{array}{c}\text{Case } (m_1,m_2)\neq(1,1) \text{ and}\\ 2m_1^2+2m_2^2+2-5m_1-5m_2-5m_1m_2=0\end{array}}
$$

$$
\begin{aligned}
m_1 &= \tfrac{(5+3\sqrt{3})(s-2+\sqrt{3})(s-7+4\sqrt{3})}{(1-s)(1+s)}\\
m_2 &= \tfrac{(5+3\sqrt{3})(s+2-\sqrt{3})(s+7-4\sqrt{3})}{(1-s)(1+s)}
\end{aligned}
$$

$$2-\sqrt{3} < |s| < 1 \text{ or } |s| < 7-4\sqrt{3}.$$

$$Z' = \begin{pmatrix} F & * \\ 0 & * \end{pmatrix} Z$$

$$\begin{aligned}-16(-s^2-7+4\sqrt{3})(x^2+1)^2F[1,1] &= -8(1-6s+s^2)(-2+\sqrt{3})x^3\\ &\quad+16(-s-3+2\sqrt{3})(3s-3+2\sqrt{3})x^2\\ &\quad-8(\sqrt{3}+2)(-s^2+56\sqrt{3}+24s\sqrt{3}-97-42s)x\end{aligned}$$

$$\begin{aligned}-16(-s^2-7+4\sqrt{3})(x^2+1)^2F[1,2] &= -8(-s-3+2\sqrt{3})(3s-3+2\sqrt{3})x^3\\ &\quad-16(1-6s+s^2)(-2+\sqrt{3})x^2\\ &\quad+8(-s-3+2\sqrt{3})(3s-3+2\sqrt{3})x\\ &\quad+32s^2+224-128\sqrt{3}\end{aligned}$$

$$\begin{aligned}-16(-s^2-7+4\sqrt{3})(x^2+1)^2F[2,1] &= -8(-s-3+2\sqrt{3})(3s-3+2\sqrt{3})x^3\\ &\quad+16(\sqrt{3}+2)(-s^2+56\sqrt{3}+24s\sqrt{3}-97-42s)x^2\\ &\quad+8(-s-3+2\sqrt{3})(3s-3+2\sqrt{3})x\\ &\quad-32s^2-224+128\sqrt{3}\end{aligned}$$

$$\begin{aligned}-16(-s^2-7+4\sqrt{3})(x^2+1)^2F[2,2] &= -8(\sqrt{3}+2)(-s^2+56\sqrt{3}+24s\sqrt{3}-97-42s)x^3\\ &\quad-16(-s-3+2\sqrt{3})(3s-3+2\sqrt{3})x^2\\ &\quad-8(1-6s+s^2)(-2+\sqrt{3})x\end{aligned}$$

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On a constructive approach to the solution of dynamical systems

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Abstract. A series of constructive results on dynamical systems governed by systems of nonlinear ODEs are presented. We use the fact that a large class of dynamical systems can be reshaped in a form involving only polynomial, or more generally, quasi-polynomial (QP) nonlinearities in their vector fields. Such QP-systems can be cast into a standard representation that is form invariant under a certain group of transformations. Among these transformations, there always exists one particular that transforms a given QP-system into a canonical form involving only homogeneous quadratic nonlinearities and entirely characterized by a fundamental matrix. The use of that canonical form leads us to several results on the integrability conditions of these systems and on the Taylor expansion of the general solution. We present some arguments for a conjecture relating the asymptotic solutions of the canonical form to another linear equation in an infinite vector space. A perturbative scheme is described for solving the latter. Its convergence will be studied in further works.

Introduction

It is a great pleasure for me to contribute to this volume dedicated to our friend Jean Thomann. Along with his warm simplicity and kindness, he is one of the few contemporary mathematicians who are trying to build an effective bridge between pure and applied, or more precisely, applicable mathematics. His fight to tame divergent series and to construct efficient algorithms implementing the most important results in the resummation theory pervades this book. The orientation of the work I am presenting here has strong connections with his constructive approach of mathematics and has been frequently enriched by discussions with Jean. I hope this dialogue will continue for many years.

It is well known that for systems of nonlinear ODEs such as the so-called autonomous dynamical systems

$$\dot{x}_i = f_i(x_1, \ldots, x_n) \tag{1}$$

(where the dot denotes the time derivative, the index i runs from 1 to n, the dependent variables x_i are real, the functions f_i are enough regular to ensure the existence and unicity of solutions, and the boundary conditions are given at the initial time), there does not exist any general algorithm for finding the general solution. This is in strong contrast with the case of linear autonomous dynamical systems

One reason for this is the infinite diversity of possible functional forms of the f_i in equation (1), in contrast with the unicity, up to the choice of a constant matrix, of the linear functions defining a linear dynamical system. Moreover, the nonlinearity of the equations entails a factorial explosion in the coefficients of the time Taylor series for the solutions: Each order k coefficient involves, indeed, a sum of $k!$ terms which depend on the specific form of the f_i. Hence, one should not be surprised by the fact that in most cases it is impossible to find a closed form structure for the general k order coefficient, and that the recursion relations between these coefficients are as difficult to solve as the original ODEs! Consequently, no information is available on the convergence of the Taylor series representing the solutions, and resummation techniques in order to find more properties of the solutions, such as their asymptotic behaviours, are impossible to implement since this, generally, requires the knowledge of the general coefficient of the series.

This is why a geometrical approach in the phase space, based on invariant properties of the phase portrait under homeomorphisms and diffeomorphisms, has been developed since H. Poincaré with the success we know. For low dimensions of the phase space and of the parameters space, complete classification of bifurcations, and the associated normal form equations have been obtained. Characterization of chaotic regimes have been obtained thanks to the combination of topological and stochastic methods. Some of these results have even been generalized to certain classes of nonlinear partial derivative equations, more precisely, those PDEs that are extensions of dynamical systems, taking into account effects due to spatial inhomogeneities like diffusion. In turn, these developments have led physicists to the elucidation, in terms of bifurcations and of resonances, of the mechanisms that are behind the emergence of spatio-temporal structures in such systems.These results have been proved fundamental for the understanding of the universal phenomenon of auto-structuration that pervades our world from the astrophysical to the biological phenomena.

Unfortunately, most of these methods become untractable in high dimensions (usually, larger than 4) . This limitation makes the qualitative approach almost useless for the sake of the analysis of realistic dynamical systems in physics, chemistry, biology and engineering. In these fields of applications, the practitioners make even no attempt to use the theoretical phase space methods, they directly perform numerical integration using efficient discretisation schemes. However, for some purposes, this approach lacks of depth. The prediction of the bifurcation values of the control parameters, and of the asymptotic behaviour of the solutions (e.g., the stability of attractors) needs an enormous amount of numerical computation times for large realistic systems. These questions could be more efficiently handled by the means of deeper algorithms generated by new theoretical considerations. This state of the matter imposes a new quest for more constructive results and our small collaboration group scattered in the world is making some attempts to contribute to this goal. Most of us are physicists and this, as a matter of fact, is reflected in the informal presentation of this work. Since our methods are not widely known, I report here the chronological steps which led us through more than a decade to the results presented in this article. At the end of this work, I am daring some conjectural ideas about a possible new approach to the asymptotic solution of a class of nonlinear dynamical systems.

The quasi-polynomial approach

It all started from the discovery in 1988 [1], [2] that a wide class of dynamical systems with polynomial or, even, quasi-polynomial (i.e., with monomials involving non-integer powers of the dependent variables, or quasi-monomials) functions f_i, could be cast into a useful standard shape, the so-called quasi-polynomial (QP) representation:

$$\dot{x}_i = x_i \sum_{j=1}^{N} A_{ij} \prod_{k=1}^{n} x_k^{B_{jk}} \qquad \text{for } i = 1, \ldots, n \tag{1}$$

where N refers to the numbers of monomials in the right hand side, and A and B are real and constant rectangular matrices. The presence of the factor x_i in front of the right hand side of this equation is essential and constrains the definition of both matrices. However, it does not restrict the generality of our purpose as any polynomial or quasi-polynomial can be represented in that form. It also underlines the importance of the logarithmic time derivative of the functions x_i which plays a fundamental role in this approach. Other authors made independently the same statement and obtained theoretical results in the fields of modeling chemical reactions and ecological systems[3], [4], [5]. These works were mainly concerned with models describing complex networks of interacting entities and they were essentially devoted to the study of the stability properties of the solutions.

A fundamental feature of the above QP representation of a dynamical system is its covariance under the group of quasi-monomial transformations of the dependent variables:

$$x_i = \prod_{k=1}^{n} \tilde{x}_k^{C_{ik}} \qquad \text{for } i = 1, \ldots, n \tag{2}$$

where the matrix C is any invertible, constant, real square matrix. In order to avoid unrelevant discussions about the existence of the inverses of these transformations and their differentiability we limit our scope to systems and to initial conditions such that the solutions remain in the positive cone. However, it can be proven that most of the following results continue to be valid for systems not fulfilling this restriction. Under these transformations the system (1) becomes:

$$\dot{\tilde{x}}_i = \tilde{x}_i \sum_{j=1}^{N} \tilde{A}_{ij} \prod_{k=1}^{n} \tilde{x}_k^{\tilde{B}_{jk}} \quad \text{for } i = 1, \ldots, n \tag{3}$$

with the following rule of transformations for the matrices A and B:

$$\tilde{A} = C^{-1} A \tag{4}$$

and

$$\tilde{B} = BC \tag{5}$$

where the products are matrix products. The form of (3) clearly exhibits the covariance of the QP equations under the quasi-monomial group. Let us stress that:

$$\tilde{B}\tilde{A} = BA, \tag{6}$$

i.e., the matrix BA is an invariant of the quasi-monomial group of transformations. This means that the whole set of QP-systems is divided into equivalence classes labeled by such $N \times N$ matrices. The fundamental matrix BA is related to the existence of canonical forms as we shall see later.

Chronologically, we first focused our work on integrability conditions and on a constructive research of invariants of the motion. We showed with A. Goriely [6] that for any system of form (1) there exists $(n - r)$ (where r is the rank of the matrix A) quasi-monomial invariants of the motion. A dual result is obtained when the rank of the matrix B is r: There exists a transformation of the dependent variables which decouples the system into a closed set of r nonlinear ODEs of QP type and $(n - r)$ linear ODEs with time varying coefficients depending on the solution of the r first nonlinear equations.

Next, we proved that, under a particular quasi-monomial transformation (as defined above) of the dependent variables, any QP-system can be brought to a canonical form, the so-called Lotka–Volterra system of ODEs [1], [2], [7]:

$$\dot{x}_i = x_i \sum_{j=1}^{N} M_{ij} x_j \quad \text{for } i = 1, \ldots, N \tag{7}$$

where N is the number of monomials of the original QP system (1) and the square $N \times N$ matrix M is the matricial product BA of its two matrices B and A. There also exists a second canonical form dual of the above one, which until now has been much less studied [1]. It is also characterized by the matrix BA. The passage to the Lotka–Volterra form is most easily shown in the particular case where the number N of monomials is equal to the dimension n of the system (1), the so-called square QP-system. In this case, and if the matrix B is not singular, the particular quasi-monomial transformation (2) in which the matrix C is equal to B^{-1}, leads to

$$\tilde{A} = BA \tag{8}$$

and

$$\tilde{B} = I \tag{9}$$

where I is the identity matrix. The resulting transformed equation is now:

$$\dot{x}_i = x_i \sum_{j=1}^{N} (BA)_{ij} x_j \quad \text{for } i = 1, \ldots, N \tag{10}$$

which is exactly the Lotka–Volterra form announced in (7) in which the matrix M is the invariant matrix BA. Notice that we omitted the tilde accent on the new variables x_i. In the more general situations in which N is different from n, the Lotka–Volterra form is also obtained. When $N < n$, the rank of the matrix B is at most N. Hence, the system can be decoupled into a closed set of N nonlinear QP equations plus $(n - N)$ linear equations. The new B matrix of the reduced N dimensional system is a square matrix of rank N and we are brought to the previous case, the square QP-system. In the case where $N > n$, we perform an embedding of the original system (1) by adding $(N - n)$ trivial equations

$$\dot{x}_i = 0 \quad \text{for } i = n + 1, \ldots, N \tag{11}$$

with initial conditions $x_i = 1$ for all i from $n + 1$ to N. This embedding transforms the system into a square QP-system of dimension N with a square matrix B which is built by adding $N - n$ columns of arbitrary entries to the original rectangular B matrix of the system (1), and which, by construction, is of maximal rank N. The new square A matrix of the embedded system is constructed from the original A matrix of (1) by adding $N - n$ lines of zeroes. Hence, the system can be transformed via the quasi-monomial transformation (2) with a matrix $C = B^{-1}$ to the canonical Lotka–Volterra form in N dimensions with the same matrix BA as before.

The fact that the transformation leading to the canonical form is a diffeomorphism reduces the analysis of the solutions of general QP-systems to those of the corresponding Lotka–Volterra canonical form. This result paved the way to the transfer of the whole corpus of knowledge related to the latter equation which is enormous (see [3], [8], [9], [10] and the references therein for a starting point). This huge task is far from being achieved but should be pursued since it would bring a flow of new results for the more general class of QP-systems which is ubiquitous in scientific and technical applications. Furthermore, we have proved that many systems that do not belong to that class may be embedded into the QP class [11] and, consequently, can be brought to the Lotka–Volterra format.

Using the Lotka–Volterra format, three important works in the field of integrability and its related questions have been realized. A. Goriely [7], [12] showed that the

Painlevé analysis (or Painlevé test) for detecting integrability conditions of systems of nonlinear ODEs finds its natural framework in the QP-representation. A dynamical system has the Painlevé property if its general solution does not present any movable critical singularity in the complexified time variable, i.e., any critical singularity whose location depends on the initial conditions. It is shown that this property is necessary for a system to be integrable but the sufficiency is still unproven and certainly is wrong below three dimensions. A. Goriely constructed a general algorithm for the Painlevé test that checks the existence of this property in any QP-system. This algorithm is based on the Lotka–Volterra format of the system. He was even able to generalize the above result to certain systems presenting algebraic singularities in the complex time.

The second work is that of A. Figueiredo [13], [14]. It is devoted to the integrability of QP-systems in terms of invariants of the motion belonging to the set of quasi-polynomials functions of the dependent variables. This is a difficult question since, in contrast with the vector space of true polynomials, the quasi-monomials do not constitute a basis of a vector space. He proved a fundamental decomposition theorem showing that any QP invariant of such systems can be decomposed into homogeneous, irreducible, true polynomial that are semi-invariants (or Darboux polynomials) of the associated Lotka–Volterra form. This solved the question and led to an exhaustive and efficient algorithm for finding and constructing such invariants.

Both previous works have been implemented into two Computer Algebra softwares written in the language MAPLE, respectively NODES [24] and [QPSI] [25].

Finally, the work of B. Hernández-Bermejo [15] concerns conditions on QP-systems to be Hamiltonian or more generally to have a Poisson structure. This led him to determine integrability conditions and algorithms for constructing invariants of the motions for these systems. He studied the link between the canonical transformations and the quasi-monomial transformations and showed how the latter, combined with the Darboux theorem on Poisson systems, leads to the determination of the Hamiltonian structure of the QP-systems. This opens the way to the application to these systems of the many results known on Hamiltonian dynamics (torus topology of the invariant surfaces, KAM theorem about their destruction under perturbations of the Hamiltonian etc.).

When a system is not integrable, one turns to stability analysis in order to study the asymptotic behaviour of the solutions. In the common practice of nonlinear dynamics, this kind of analysis is performed in the near vicinity of the fixed points. The linear stability of these points is governed by the eigenvalues of the linearized vector field. However, this approach fails when one tries to predict the fate of solutions in larger domains around the fixed points or near other invariant sets such as cycles or strange attractors. In these cases, the nonlinearity plays a fundamental role and cannot be neglected.

One approach relies on the search of a Lyapunov function which, roughly said, plays a role similar to the potential energy for mechanical systems. Such a function of the dependent variables $V(x)$ must be non-negative in its definition set, should vanish at a given fixed point and must possess a non-positive (or negative) derivative along the flow of the system. The existence of such a function, then, guarantees the stability (or asymptotic stability) of the orbits in a finite domain around the fixed point. However, this idyllic picture breaks down when one tries to find such a function! Generally, there is no algorithm for doing this, with one exception: the Lotka–Volterra system [8], [9], [10]! This is why the practitioners almost never try to use this approach when handling other systems since it essentially relies on intuition, a typically non-algorithmic tool. However, the QP-representation (1) provides a bridge between most dynamical systems used in scientific models and the Lotka–Volterra systems. Several works have taken profit of this opportunity and have led to important theoretical results [4], [5], [17] and to powerful algorithms [18] for studying the nonlinear stability and the attraction basin of the attracting invariant sets of most dynamical systems of practical interest.

Let us now turn towards other methods for studying non-integrable systems.

Series expansions

Another constructive approach to the general solution of non-integrable dynamical systems is provided by several types of series expansions, the Poincaré–Dulac normal form and the Taylor series. The first one [19], amounts to search a transformation that exactly linearizes the original system (1). In order to achieve this program, one must first extract the linear part of the vector field around a given fixed point. Then, one tries to find a diffeomorphism of the dependent variables that linearizes the system obeyed by the new dependent variables while keeping exactly the same linear part as in the original variables up to a linear conjugation. This means that the local properties of the phase portrait remain unaltered by the transformation.

It is then shown that obstructions to this program appear whence the eigenvalues $(\lambda_1, \dots, \lambda_n)$ of the linear part of the vector field satisfy resonance conditions:

$$m_1\lambda_1 + \cdots + m_n\lambda_n = \lambda_i \tag{12}$$

for sets of non-negative integers $(m_1, \dots, m_n)$ satisfying $(m_1 + \cdots + m_n) > 1$, and for some values of i among $1, \dots, n$. For any such resonance there appears, in the transformed system obeyed by the new dependent variables, a nonlinear monomial in these variables of degree $(m_1 + \cdots + m_n) > 1$. The coefficients of these monomials depend on the coefficients of the Taylor expansion of the "linearizing" transformation which, in turn, satisfy a complicated nonlinear recursion relation.

For most dynamical systems, these recursion relations are as difficult to solve as the differential system itself. As in the case of the Taylor series in powers of the time variable for the general solution, there is a factorial explosion of terms in the general k order coefficient. This seemingly excludes the possibility of finding a closed form expression for these coefficients. As a consequence, the convergence rate of these series is not available except in particular conditions for which convergence theorems do exist. However, it is expected that many of these series should diverge since their coefficients depends on products of factors of type $(m_1\lambda_1 + \cdots + m_n\lambda_n - \lambda_i)^{-1}$. As the series involve sums over all the non-negative values of the m_i, these denominators may take values that are arbitrarily near zero, even if not strictly vanishing, and this generally causes the divergence of the series.

For non-resonant systems the general structure of these coefficients has been derived [16] by S. Louies in his PhD thesis by using the Lotka–Volterra format of the systems. He has been able to find explicit expressions for these general coefficients and to exploit them in order to find resummation schemes using Padé approximants. These results are identical to those obtained with the concepts of mould and comould by J. Ecalle [20], [21] through a very different approach.

I shall not give here the expressions of these coefficients since they are related to the Taylor coefficients which will be studied in details in the sequel. When all the eigenvalues of the linear part of the system vanish, it can be shown that the series expansion of the Poincaré–Dulac normalizing transformation coincides with the time Taylor series for its solution. Furthermore, it is also easily shown that a Lotka–Volterra system of dimension n with a diagonal linear contribution is equivalent to a similar system of dimension $n+1$ without any linear term. Hence, one can focus on the Taylor series for the Lotka–Volterra form (7) without any loss of generality. A compact expression for the general coefficient can be derived [7] in the following way. The Taylor series in time for the solution $x_i(t)$ of the Lotka–Volterra equation is defined as

$$x_i(t) = \sum_{k=0}^{\infty} c_i(k)\frac{t^k}{k!} \qquad \text{for } i = 1, \ldots, N \tag{13}$$

The coefficient $c_i(k)$ is computed by performing the k order time derivative of $x_i(t)$ at time $t = 0$. This is readily done by iterating the time derivative while using recursively the Lotka–Volterra equation (7). The result is amazingly simple:

$$c_i(k) = x_i(0) \sum_{i_1=0}^{N} \cdots \sum_{i_k=0}^{N} M_{ii_1}\left(M_{ii_2} + M_{i_1i_2}\right)\ldots\left(M_{ii_k} + M_{i_1i_k} + \cdots + M_{i_{k-1}i_k}\right) x_{i_1}(0)\ldots x_{i_k}(0)$$

Going back to the original variables of the QP-equation (1), we get for the order k coefficient $C_i(k)$ of their time Taylor series:

$$C_i(k) = x_i(0) \sum_{i_1=0}^{N} \cdots \sum_{i_k=0}^{N} A_{ii_1}\left(A_{ii_2} + M_{i_1i_2}\right)\ldots\left(A_{ii_k} + M_{i_1i_k} + \cdots + M_{i_{k-1}i_k}\right) \prod_{j_1=1}^{n} x_{j_1}(0)^{B_{i_1j_1}} \cdots \prod_{j_k=1}^{n} x_{j_k}(0)^{B_{i_kj_k}}$$

where i runs from 1 to n and n is the dimension of the original QP-system.

The structure of these coefficients clearly reveals a combinatorial property related to the product $M_{ii_1}\left(M_{ii_2}+M_{i_1i_2}\right)\ldots\left(M_{ii_k}+M_{i_1i_k}+\cdots+M_{i_{k-1}i_k}\right)$. For a one dimensional Lotka–Volterra system this product would reduce to $k!M^k$. However, in more dimensions the number M is replaced by the components of the tensor M_{ij}. More precisely, if we introduce sums in the above expression over the indices $j_1,\ldots,j_k$, each one running from 1 to N, and also introduce the corresponding Kronecker deltas in order to compensate them, the above product becomes

$$\sum_{j_1=0}^{N}\cdots\sum_{j_k=0}^{N}\delta_{ij_1}\left(\delta_{ij_2}+\delta_{i_1j_2}\right)\ldots\left(\delta_{ij_k}+\delta_{i_1j_k}+\cdots+\delta_{i_{k-1}j_k}\right)M_{j_1i_1}M_{j_2i_2}\ldots M_{j_ki_k}$$

where the tensor

$$\delta_{ij_1}(\delta_{ij_2}+\delta_{i_1j_2})\ldots\left(\delta_{ij_k}+\delta_{i_1j_k}+\cdots+\delta_{i_{k-1}j_k}\right)$$

is a universal object in the sense of J.Ecalle. It appears in the solution of any Lotka–Volterra or QP-system independently of the particular forms of the matrix M, or A and B. Furthermore, this tensor is a generalization to the N dimensional case of the factorial function $k!$. Its role in the structure of the Taylor coefficient is double. It governs the type of contracted products between the k tensors M that appear in the coefficient $c_i(k)$, and also counts them. The combinatorial study of this object would certainly be of great importance both for combinatorics and for the study of these Taylor series. Another interesting consequence of having a closed expression for the general Taylor coefficient is that it gives the possibility of estimating the convergence radius of the series. Based on this, a numerical integrator of differential equations using the Taylor series may be constructed. Having the analytic expressions of the coefficients gives two advantages: A better control of the integration step which must be small with respect to the convergence radius, and the possibility of using the symbolic expressions of the coefficients which may depend on some parameters. These two aspects should greatly enhance the precision and the speed of computation. A first (rough!) prototype of such a symbolic-numerical program exists in the above mentioned software NODES (Nonlinear ODEs Solver).

Another way of looking at these series is by considering them as defining a new class of special functions which, in view of the combinatorial analogy discussed above, presents some analogy with the large class of the hypergeometric functions. This is by itself a whole domain that should be developed per se and which is related to the question of the summability of the series and of the asymptotics of the functions that are defined by these series. This remark leads us to the next section which precisely is concerned with this questions.

Some speculations about asymptotics

Let us start by putting the Lotka–Volterra system (7) in a more operatorial form. We write it as a linear equation in an infinite vector space for the vector

$$\psi(\vec{r};t) = \prod_{i=1}^{N} x_i(t)^{r_i} \tag{14}$$

where the components of $\vec{r}$, $r_1, \ldots, r_N$, run on all the non-negative integers. This is the well known Carleman embedding [22], [23] performed for the Lotka–Volterra canonical form of our original system (1).

The time derivative of ψ can easily be obtained from the Lotka–Volterra equation itself, and we get:

$$\partial_t \psi(\vec{r};t) = \sum_{i=1}^{N}\sum_{j=1}^{N} r_i M_{ij} D_j \psi(\vec{r};t) \tag{15}$$

where D_i is the finite displacement operator defined by:

$$D_i f(r_1, \ldots, r_i, \ldots, r_N) = f(r_1, \ldots, r_i + 1, \ldots, r_N) \tag{16}$$

The original variable x_i can be retrieved from the ψ function by posing in it $r_j = 0$ for all $j \neq i$ and $r_i = 1$. In order to have a complete equivalence between (15) and the Lotka–Volterra equation we must impose initial conditions of the form

$$\psi(\vec{r};t=0) = \prod_{i=1}^{N} x_i(0)^{r_i} \tag{17}$$

The general solution of this equation is obtained through the action of the exponential of the operator $\sum_{i=1}^{N}\sum_{j=1}^{N} r_i M_{ij} D_j$ on the initial condition given by (17). The successive powers of that operator acting on the above initial condition provide also the Taylor coefficients for the expansion in power of the time t of the function $\psi(\vec{r};t)$. Their expressions is readily found and has the same general combinatorial structure as found above. Since nothing can be added to our previous comments on these coefficients we do not give here their expression.

At this stage one could be tempted to say that this representation in an infinite vector space does not bring much novelty into the problem of effectively solving nonlinear ODEs. However, some speculations may be here of interest. Although these ideas should be cautiously considered, I present them to the discussion since they could open some new paths. Let us consider the following linear equation in the same infinite vector space as above:

$$\partial_t \phi(\vec{r};t) = \sum_{i=1}^{N}\sum_{j=1}^{N} r_i M_{ij} r_j \phi(\vec{r};t) - \alpha^2 \sum_{i=1}^{N}\sum_{j=1}^{N} D_i M_{ij} D_j \phi(\vec{r};t) \tag{18}$$

Let us denote by G the linear operator acting in the right hand side on function ϕ. Consider now the one-parameter group generated by $\sum_{l=1}^{N} D_l$ with parameter α, and let us transform the function $\phi(\vec{r}; t)$ as follows:

$$\phi(\vec{r}; t; \alpha) = e^{\alpha \sum_{l=1}^{N} D_l} \phi(\vec{r}; t) \tag{19}$$

It obeys an equation obtained by transforming the linear operator G in the usual way:

$$\partial_t \phi(\vec{r}; t; \alpha) = e^{\alpha \sum_{l=1}^{N} D_l} G e^{-\alpha \sum_{k=1}^{N} D_k} \phi(\vec{r}; t; \alpha) \tag{20}$$

A simple calculation involving iterated commutators between $\sum_{l=1}^{N} D_l$ and G leads to:

$$\partial_t \phi(\vec{r}; t; \alpha) = \sum_{i=1}^{N} \sum_{j=1}^{N} r_i M_{ij} r_j \phi(\vec{r}; t; \alpha) + 2\alpha \sum_{i=1}^{N} \sum_{j=1}^{N} r_i M_{ij} D_j \phi(\vec{r}; t; \alpha) \tag{21}$$

The second linear operator occurring in this equation is identical, up to a constant factor 2α, to the operator of the equation (15). However, one should stress that this is only true for symmetrical matrices M, since in equation (18) only the symmetrical part of the matrix M is retained. Moreover, the condition of vanishing diagonal entries $M_{ii} = 0$ for all i running from 1 to N has been used to pass from equation (18) to equation (21). *This restricts our scope to QP-systems whose matrices A and B are such that their product BA is a symmetrical matrix with vanishing diagonal entries.* We now rescale the time by:

$$\tau = 2\alpha t \tag{22}$$

Under this rescaling, we get:

$$\partial_\tau \phi(\vec{r}; \tau; \alpha) = \frac{1}{2\alpha} \sum_{i=1}^{N} \sum_{j=1}^{N} r_i M_{ij} r_j \phi(\vec{r}; \tau; \alpha) + \sum_{i=1}^{N} \sum_{j=1}^{N} r_i M_{ij} D_j \phi(\vec{r}; \tau; \alpha) \tag{23}$$

We can now state a speculative hypothesis in the form of the following question. Would the solutions of this last equation converge for large values of α towards the asymptotic (large τ) solutions of equation (15), where the variable t has to be interpreted as τ? Or, put in an other way, if we consider the first term in the right hand side of (23) as a small perturbation for large α, would the leading term of the perturbation series be the asymptotic solution of equation (15)? The interplay between the limits of large α and large τ is the following. We take the limit of large α in such a way that the ratio $\frac{\tau}{2\alpha}$, which is equal to t, is kept constant. Hence, this constrained limit implies a large τ limit, i.e. the asymptotic limit.

More precisely, a perturbation scheme for (23) can be set up by expanding $\phi(\vec{r}; \tau; \alpha)$ in powers of the small parameter $\frac{1}{\alpha}$, $\phi(\alpha) = \phi_0 + \frac{1}{\alpha}\phi_1 + \frac{1}{\alpha^2}\phi_2 + \cdots$.

Hence, the functions ϕ_k are solutions of the infinite system

$$\partial_\tau \phi_0 = \sum_{i=1}^{N} \sum_{j=1}^{N} r_i M_{ij} D_j \phi_0 \tag{24}$$

$$\partial_\tau \phi_k = \sum_{i=1}^{N} \sum_{j=1}^{N} r_i M_{ij} D_j \phi_k + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} r_i M_{ij} r_j \phi_{k-1} \tag{25}$$

for all the integers $k \geq 1$. As expected, the first equation precisely is the equation (15) we want to solve asymptotically. Its solution would be the leading term of the perturbation series if it could be proven that it converges. However, equation (23) is equivalent to equation (18) which looks easier to solve. A perturbative solution can be obtained by first rescaling the time t in (18)

$$\tilde{t} = \alpha^2 t \tag{26}$$

Then, $\phi\left(\vec{r};\tilde{t}\right)$ is expanded in powers of $\frac{1}{\alpha^2}$, $\phi = \tilde{\phi}_0 + \frac{1}{\alpha^2}\tilde{\phi}_1 + \frac{1}{\alpha^4}\tilde{\phi}_2 + \cdots$, where the functions $\tilde{\phi}_k$ obey the infinite system

$$\partial_{\tilde{t}} \tilde{\phi}_0 = -\sum_{i=1}^{N} \sum_{j=1}^{N} D_i M_{ij} D_j \tilde{\phi}_0 \tag{27}$$

$$\partial_{\tilde{t}} \tilde{\phi}_k = -\sum_{i=1}^{N} \sum_{j=1}^{N} D_i M_{ij} D_j \tilde{\phi}_k + \sum_{i=1}^{N} \sum_{j=1}^{N} r_i M_{ij} r_j \tilde{\phi}_{k-1} \tag{28}$$

for all positive integers $k \geq 1$. The first equation of the above system is readily solved by using the Fourier transform over the variable $\vec{r}$. Its solution is given by

$$\tilde{\phi}_0\left(\vec{r};\tilde{t}\right) = \int d^N r' K\left(\vec{r} - \vec{r}';\tilde{t}\right)\tilde{\phi}_0\left(\vec{r}';\tilde{t} = 0\right) \tag{29}$$

where the integral kernel K is given by

$$K\left(\vec{r} - \vec{r}';\tilde{t}\right) = \frac{1}{(2\pi)^N} \int d^N k e^{-i\vec{k}\cdot\left(\vec{r}-\vec{r}'\right)} e^{-\tilde{t}\sum_{j=1}^{N}\sum_{l=1}^{N} M_{jl} e^{-i(k_j+k_l)}} \tag{30}$$

The next terms of the perturbation expansion are obtained by iterating convolution integrals over the variable $\tilde{t}$, and over $\vec{r}$, of the term $\sum_{i=1}^{N}\sum_{j=1}^{N} r_i M_{ij} r_j \tilde{\phi}_{k-1}$ and the kernel K. The last steps are the following. First, one replaces $\tilde{t}$ by its expression $\alpha^2 t$. This yields the perturbative solution of (18). Then, the transformation $e^{\alpha \sum_{l=1}^{N} D_l}$ is performed on each term of that series. This gives the expansion of the solution to equation (21) on which, after rescaling of the time (22), the limit for large α is taken, and the corrections for large α are regrouped for increasing powers of $\frac{1}{\alpha}$.

This procedure, if our initial assumption is correct, should provide the asymptotic expansion in time of the solution to equation (15) and, in turn, to the Lotka–Volterra equation for a symmetrical matrix M with vanishing diagonal entries. The asymptotic solution to the initial QP equation (1) is then recovered by performing the inverse of the QM transformation (2).

I insist, at this stage the above scheme is purely speculative and preliminary. The convergence of the perturbative expansions involved in it should be carefully studied. Moreover, and in parallel, non-trivial examples should be worked out along these lines and compared to numerical integration results.

Conclusion

I did not mention other advances made recently in the QP framework, and I briefly comment on them. Since the Lotka–Volterra canonical form involves only homogeneous quadratic nonlinear terms in its vector field, it is possible to merge it in a larger second order system describing geodesics in a N-dimensional curved space. Generally, this space is not metrical and is characterized by an affine connection which linearly depends on the matrix M. With P. Bieliavsky, we recently have generalized this approach to the whole class of systems of ODEs with homogeneous quadratic vector fields, and have proved that the only cases in which a metrics can be derived correspond to a class of completely integrable systems. However, when a metrics does not exist, there remain many interesting properties related to the Riemann–Christoffel tensor of the space. Vanishing conditions of this tensor or of its covariant derivatives are related to integrability properties of the related ODEs system. Moreover, this geodesic representation is particularly well fitted to the calculation of the geodesic separation and the description of chaotic regimes (e.g., Lyapunov exponents). To conclude, I hope this presentation of the potentialities of the QP representation have been enough convincing to stir up the interest of the mathematicians working in the fields of nonlinear dynamics, asymptotic resummation and combinatorics.

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Sur la “solution analytique générale” d’une équation différentielle chaotique du troisième ordre

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Résumé. Même si elle est non-intégrable, une équation différentielle peut néanmoins admettre des solutions particulières globalement analytiques. Sur l’exemple du système dynamique de Kuramoto et Sivashinsky, génériquement chaotique et d’un grand intérêt physique, nous passons en revue diverses méthodes, toutes fondées sur la structure des singularités, permettant de caractériser la solution analytique qui dépend du plus grand nombre possible de constantes d’intégration.

Abstract. Even if it is nonintegrable, a differential equation may nevertheless admit particular solutions which are globally analytic. On the example of the dynamical system of Kuramoto and Sivashinsky, which is generically chaotic and presents a high physical interest, we review various methods, all based on the structure of singularities, allowing us to characterize the analytic solution which depends on the largest possible number of constants of integration.

Mots clefs : ondes solitaires, équations de Briot et Bouquet, Nevanlinna, algorithme LLL, équation de Kuramoto et Sivashinsky, équation de Ginzburg et Landau complexe cubique unidimensionnelle, genre, troncature.
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1 Situation du problème

D’après un résultat classique [28], un système différentiel ne peut présenter un comportement chaotique que si son ordre égale au moins trois. D’autre part, malgré l’opposition intuitive entre chaos et analyticité, il n’est pas interdit à un système génériquement chaotique de posséder des solutions particulières globalement analytiques, la seule impossibilité étant que le nombre de constantes d'intégration ne peut égaler l'ordre. Le but de cet article est de passer en revue les diverses méthodes connues permettant de caractériser (idéalement d'exhiber) cette solution analytique, forcément particulière, qui dépend du nombre maximal possible de constantes d'intégration. Convenons de l'appeler *solution analytique générale*, par opposition à la *solution générale* dont l'expression analytique n'existe pas.

L'exemple physique optimal pour illustrer ces méthodes est à notre avis l'équation de Ginzburg et Landau cubique unidimensionnelle (CGL3)

$$i\psi_t + p\psi_{xx} + q|\psi|^2\psi - i\gamma A = 0, \quad pq\gamma \neq 0,\ (\psi, p, q) \in \mathcal{C},\ \gamma \in \mathcal{R}, \quad (1.1)$$

pour les raisons suivantes,

1. elle est génériquement chaotique [21],

2. elle admet une limite intégrable $\Im(p) = \Im(q) = \gamma = 0$ (équation de Schrödinger non-linéaire (NLS)) dont tous les éléments d'íntégrabilité sont connus analytiquement,

3. les phénomènes physiques qu'elle modélise sont très importants : propagation du signal dans une fibre optique [1], intermittence spatio-temporelle en turbulence, supraconductivité, etc.

Nous nous contenterons d'étudier ici une équation plus simple, celle de Kuramoto et Sivashinsky (KS), déduite de celle pour la phase $\varphi = \arg\psi$ du champ ψ de CGL3 en prenant une certaine limite [25, 19],

$$\varphi_t + \nu\varphi_{xxxx} + b\varphi_{xxx} + \mu\varphi_{xx} + \varphi\varphi_x = 0, \quad \varphi \in \mathcal{C},\ (\nu, b, \mu) \in \mathcal{R},\ \nu \neq 0. \quad (1.2)$$

Sa réduction en onde propagative

$$\varphi(x,t) = c + u(\xi), \quad \xi = x - ct, \quad (1.3)$$

$$E(u,\xi) \equiv \nu u''' + bu'' + \mu u' + \frac{u^2}{2} + A = 0, \quad (\nu, b, \mu) \in \mathcal{R},\ \nu \neq 0, \quad (1.4)$$

où A est une constante d'intégration, a également un comportement chaotique [21], et elle dépend de deux paramètres sans dimension, $b^2/(\mu\nu)$ et $\nu A/\mu^3$. Nous noterons désormais x pour ξ. Le problème à résoudre est le suivant.

Problème. Trouver l'expression explicite de la "solution analytique générale" (définition *supra*) de l'EDO (1.4).

La motivation vient de ce que cette solution (pour CGL3 et donc pour KS) est "observée" sous forme de texture bien définie, tant dans des simulations numériques que dans de véritables expériences physiques [11].

2 Suppression locale de la contribution chaotique

Comptons d’abord le nombre (nécessairement inférieur à trois) de constantes arbitraires dans la solution analytique générale. La recherche d’un comportement local algébrique au voisinage d’une singularité mobile x_0 (mobile signifie : qui dépend des conditions initiales),

$$u \sim_{x \to x_0} u_0 \chi^p, \quad u_0 \neq 0, \quad \chi = x - x_0, \tag{2.1}$$

conduit à la série de Laurent

$$u^{(0)} = 120\nu\chi^{-3} - 15b\chi^{-2} + \left(\frac{60\mu}{19} - \frac{15b^2}{76\nu}\right)\chi^{-1} + O(\chi^0), \tag{2.2}$$

d’où sont absentes deux des trois constantes arbitraires. Celles-ci apparaissent en perturbation [4],

$$u = u^{(0)} + \varepsilon u^{(1)} + \varepsilon^2 u^{(2)} + \cdots, \tag{2.3}$$

où le petit paramètre ε ne figure pas dans l’EDO (1.4). L’équation linéarisée en $u^{(0)}$

$$\left(\nu\frac{\mathrm{d}^3}{\mathrm{d}x^3} + b\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mu\frac{\mathrm{d}}{\mathrm{d}x} + u^{(0)}\right)u^{(1)} = 0, \tag{2.4}$$

est alors du type de Fuchs au voisinage de $x = x_0$, avec pour équation indicielle ($q = -6$ désigne le degré de singularité du premier membre E de (1.4))

$$\lim_{\chi \to 0} \chi^{-j-q}(\nu\partial_x^3 + u_0\chi^p)\chi^{j+p} \tag{2.5}$$

$$= \nu(j-3)(j-4)(j-5) + 120\nu = \nu(j+1)(j^2 - 13j + 60) \tag{2.6}$$

$$= \nu(j+1)\left(j - \frac{13 + i\sqrt{71}}{2}\right)\left(j - \frac{13 - i\sqrt{71}}{2}\right) = 0. \tag{2.7}$$

La représentation locale de la solution générale,

$$\begin{aligned} u(x_0, \varepsilon c_+, \varepsilon c_-) = 120\nu\chi^{-3}\{&\mathrm{Taylor}(\chi) \\ &+ \varepsilon[c_+\chi^{(13+i\sqrt{71})/2}\mathrm{Taylor}(\chi) \\ &+ \ c_-\chi^{(13-i\sqrt{71})/2}\mathrm{Taylor}(\chi)] + \mathcal{O}(\varepsilon^2)\}, \end{aligned}$$

où “Taylor” désigne des séries convergentes de χ, dépend bien de trois constantes arbitraires $(x_0, \varepsilon c_+, \varepsilon c_-)$ (l’indice de Fuchs -1 ne représente qu’une translation de x_0). Le branchement mobile dense provenant des deux indices irrationnels caractérise [31] le comportement chaotique, et l’unique moyen de l’éliminer est d’exiger $\varepsilon c_+ = \varepsilon c_- = 0$, c’est-à-dire $\varepsilon = 0$, restreignant ainsi à un seul arbitraire la partie analytique de la solution. Le problème est donc de trouver une expression compacte pour la série de Laurent (2.2).

3 Sur la suppression globale de la contribution chaotique

Par élimination de la constante mobile x_0 (donc de $x-x_0$) entre la série de Laurent (2.2) et sa dérivée, la solution analytique générale satisfait à une EDO autonome d'ordre un,

$$F(u', u) = 0, \tag{3.1}$$

que nous appellerons *sous-équation* de (1.4), puisque cette dernière en est une conséquence différentielle.

Trouver la solution analytique générale équivaut alors à trouver une sous-équation autonome d'ordre un. En effet, si l'on admet que F est polynomiale et que sa solution générale est uniforme, alors

1. (Hermite [32, Vol. II, §139]) le genre de la courbe algébrique $F = 0$ est zéro ou un,

2. (Briot et Bouquet [24, pages 58–59]) sa forme nécessaire est

$$F(u, u') \equiv \sum_{k=0}^{m} \sum_{j=0}^{2m-2k} a_{j,k} u^j {u'}^k = 0, \quad a_{0,m} = 1, \tag{3.2}$$

   où m est un entier positif et les $a_{j,k}$ sont des constantes,

3. (Briot et Bouquet) la solution est, pour le genre zéro, une fraction rationnelle de e^{ax} ou de x (a désignant une constante); pour le genre un, une fonction elliptique.

Dans les deux cas, il existe un algorithme, implanté en Maple par Mark van Hoeij [12], qui donne l'expression explicite de la solution.

Malheureusement, à notre connaissance, on ne sait pas effectuer l'élimination de $x - x_0$ entre la série de Laurent et sa dérivée qui, si elle était possible, fournirait (3.1).

4 Rappel sur la classe d'équations $P(u^{(n)}, u) = 0$

P désignant un polynôme de deux variables à coefficients complexes, la classe d'équations autonomes

$$P(u^{(n)}, u) = 0, \tag{4.1}$$

indépendantes des dérivées d'ordre 2 à $n - 1$ inclus, a été classiquement beaucoup étudiée. Soit W la classe de fonctions d'une variable complexe x

$$W := \{\text{rationnel}(x), \text{rationnel}(e^{ax}), \text{elliptique}(x)\}, \tag{4.2}$$

où a est une constante complexe. C’est en fait la classe des fonctions elliptiques et de leurs dégénérescences, car

$$\lim_{a\to 0} \text{rationnel}(e^{ax}) = \lim_{a\to 0} \text{rationnel}((e^{ax}-1)/a) = \text{rationnel}(x). \quad (4.3)$$

Les résultats connus sont les suivants.

1. (Briot et Bouquet) Pour $n = 1$, si la solution est uniforme, alors elle est dans W.

2. (Picard) Pour $n = 2$, toute solution méromorphe est dans W.

3. (Eremenko [7]) Si n est impair, si le genre de la courbe algébrique $P = 0$ est nul, et s’il existe une solution particulière méromorphe qui possède au moins un pôle, alors cette solution est dans W.

Ce troisième cas (qui se démontre par la théorie de Nevanlinna) s’applique à (1.4) pour $b = \mu = 0$, la solution est alors connue [8]

$$b = 0, \mu = 0 : \quad u = -60\nu\wp'(x - x_0, 0, g_3),\ g_2 = 0,\ g_3 = \frac{A}{1080\nu^2}, \quad (4.4)$$

$\wp$ représentant la fonction elliptique de Weierstrass, et il en existe même une extrapolation de codimension un [16]

$$\begin{aligned} b^2 - 16\mu\nu = 0 : \quad u &= -60\nu\wp' - 15b\wp - \frac{b\mu}{4\nu}, \\ g_2 &= \frac{\mu^2}{12\nu^2},\ g_3 = \frac{13\mu^3 + \nu A}{1080\nu^3}. \end{aligned} \quad (4.5)$$

Notre problème revient à extrapoler cette solution pour une valeur arbitraire de $b^2/(\mu\nu)$.

5 Solutions de codimension non nulle

Elles appartiennent toutes (du moins celles qui sont connues) à la classe W, Eq. (4.2). L’unique solution connue de codimension trois est rationnelle,

$$b = 0, \quad \mu = 0, \quad A = 0, \quad u = 120\nu(x - x_0)^{-3}. \quad (5.1)$$

On connaît six solutions de codimension deux [17, 15], toutes rationnelles en e^{kx},

$$\begin{aligned} u = \frac{5}{2}bk^2 - \frac{13b^3}{32\times 19\nu^2} + \frac{7\mu b}{4\times 19\nu} + \left(\frac{60}{19}\mu - 30\nu k^2 - \frac{15b^2}{4\times 19\nu}\right)\tau \\ - 15b\tau^2 + 120\nu\tau^3,\ \tau = \frac{k}{2}\tanh\frac{k}{2}(x - x_0), \end{aligned} \quad (5.2)$$

les valeurs permises figurant dans la Table 1.

Table 1. Les six solutions connues de codimension deux de (1.4). Elles ont toutes la forme (5.2).

$b^2/(\mu\nu)$	$\nu A/\mu^3$	$\nu k^2/\mu$
0	$-4950/19^3,\ 450/19^3$	$11/19,\ -1/19$
$144/47$	$-1800/47^3$	$1/47$
$256/73$	$-4050/73^3$	$1/73$
16	$-18,\ -8$	$1,\ -1$

Enfin, l'unique solution connue de codimension un est elliptique, cf. (4.5), et c'est une extrapolation de la dernière ligne de la Table 1.

Une belle propriété commune à toutes ces solutions est d'admettre la représentation

$$u = \mathcal{D} \operatorname{Log} \sigma, \quad \mathcal{D} = 60\nu \frac{\mathrm{d}^3}{\mathrm{d}x^3} + 15b \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{15(16\mu\nu - b^2)}{76\nu} \frac{\mathrm{d}}{\mathrm{d}x}, \tag{5.3}$$

où σ est une fonction entière dont l'EDO est facile à construire.

Avant de rechercher plus avant la solution inconnue de codimension zéro, il convient de se poser la question si elle est ou non dans la classe W, c.à.d. si elle est elliptique.

On pourrait craindre en effet que cette classe W soit insuffisante, au vu d'une EDO qui ne diffère de (1.4) que par le terme uu' [29, 26],

$$E \equiv -u''' + 9ku'' + (a_1 u - 26k^2)u' - 2a_1 k u^2 + 24k^2 u, \tag{5.4}$$

et dont la solution sort de la classe W,

$$u = \frac{12}{a_1} e^{2kx} \wp\left(\frac{e^{kx} - 1}{k} - X_0, g_2, g_3\right), \quad (X_0, g_2, g_3) \text{ arbitraires.} \tag{5.5}$$

Une solution dans cette nouvelle classe est exclue pour (1.4), car l'EDO d'ordre un résultant de l'élimination de X_0 entre (5.5) et sa dérivée,

$$(u' - 2ku)^2 - 4u^3 + e^{4kx} g_2 u + e^{6kx} g_3 = 0, \tag{5.6}$$

n'est autonome que pour la dégénérescence rationnelle $g_2 = g_3 = 0$.

6 Vers une approche numérique de la solution elliptique

À partir de l'unique élément d'information que constitue la série de Laurent (2.2), nous allons définir plusieurs approches numériques pour tester la validité de l'hypothèse elliptique (appartenance à la classe W). L'outil naturel est celui des approximants de Padé (AP) [2].

Étant donné les N premiers termes d’une série de Taylor en $x = 0$,

$$S_N = \sum_{j=0}^{N} c_j x^j, \tag{6.1}$$

l’approximant de Padé $[L, M]$ de la série est défini comme l’unique fraction rationnelle

$$[L, M] = \frac{\sum_{l=0}^{L} a_l x^l}{\sum_{m=0}^{M} b_m x^m}, \quad b_0 = 1, \tag{6.2}$$

satisfaisant à la condition

$$S_N - [L, M] = \mathcal{O}(x^{N+1}), \quad L + M = N. \tag{6.3}$$

L’extension aux séries de Laurent ne présente pas de difficulté. En particulier, les AP sont exacts sur les fractions rationnelles dès que L et M sont suffisamment grands.

Le calcul préalable d’un grand nombre (environ 200) de termes de la série de Laurent se fait par un algorithme linéaire, la seule difficulté, aisée à surmonter, étant de ne jamais engendrer de termes qui s’avéreront inutiles. Il importe également de calculer les c_j, puis les (a_l, b_m), sur $\mathcal{Q}$, jamais sur $\mathcal{R}$, à cause d’importantes compensations. C’est pour cela que nous avons dû écarter le programme *pade* de *Numerical recipes* [27], qui engendre des doublets de Froissart [9, 10] indésirables (couples d’un zéro et d’un pôle très proches, ces couples se répartissant sur un cercle sans définir de coupure comme ce serait le cas s’il ne s’agissait pas d’erreurs numériques). La fonction *pade* du langage formel Maple [22] est en revanche tout à fait adaptée.

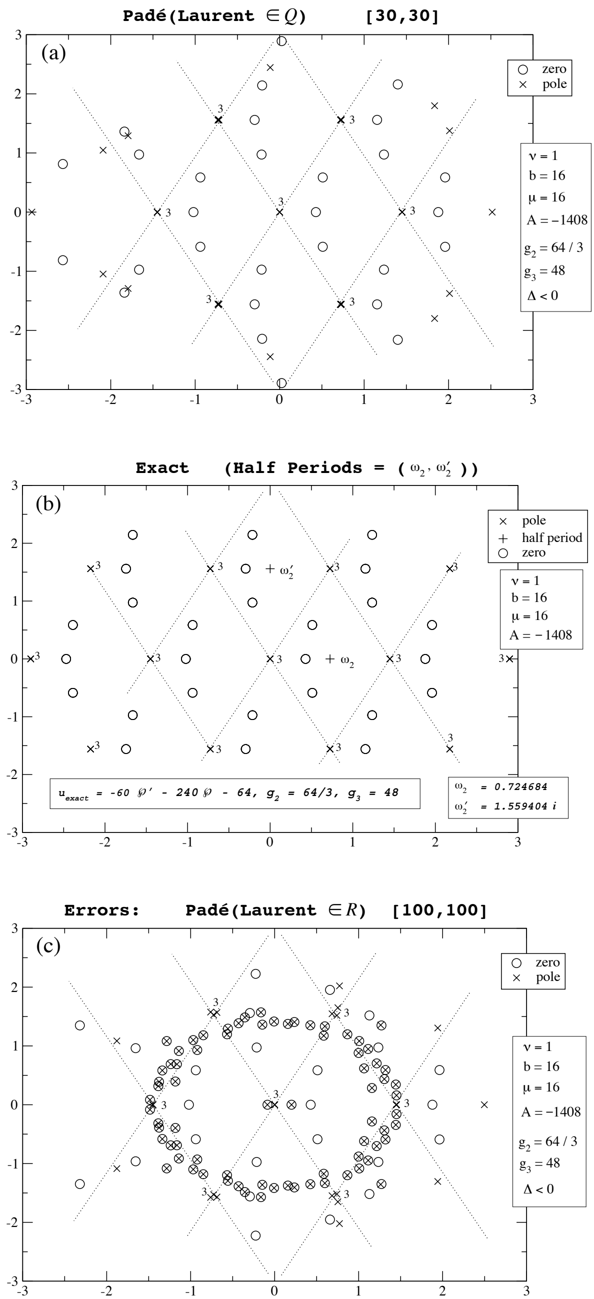
Remarque. Il est possible d’éviter l’étape de calcul des AP et d’obtenir directement et rapidement [30] les valeurs numériques des zéros et des pôles du Padé $[L, M]$.

6.1 Premier test d’ellipticité

La structure des pôles et des zéros des AP de (2.2) caractérise la classe de fonctions à laquelle appartient sa somme inconnue.

Une somme elliptique est caractérisée par une double infinité de pôles et de zéros situés aux nœuds d’un réseau doublement périodique, une somme rationnelle en e^{ax}, avec a complexe, par une simple infinité de pôles et de zéros, une somme rationnelle en x par un nombre fini de pôles et de zéros, etc. C’est pour tester les structures doublement périodiques que l’on a besoin d’un grand nombre de termes dans la série de Laurent.

Pour tous les cas de solution connue ((5.2), (4.5)), en choisissant pour les constantes fixes (ν, b, μ, A) des valeurs rationnelles, le résultat est conforme aux espérances, voir Fig. 1. Pour plusieurs cas génériques de valeurs de $b^2/(\mu\nu)$ (par exemple $b^2/(\mu\nu) = 4, 8$, Fig. 2), nous trouvons une solide indication numérique que les textures observées sont toutes doublement périodiques, ce qui est la signature des fonctions elliptiques. Dans tous les cas, connus et inconnus, il est aisé de compter le nombre (commun) de zéros et de pôles par période, et les calculs indiquent que ce nombre serait égal à trois.



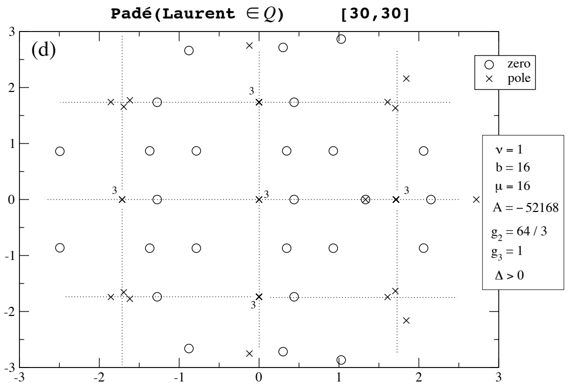
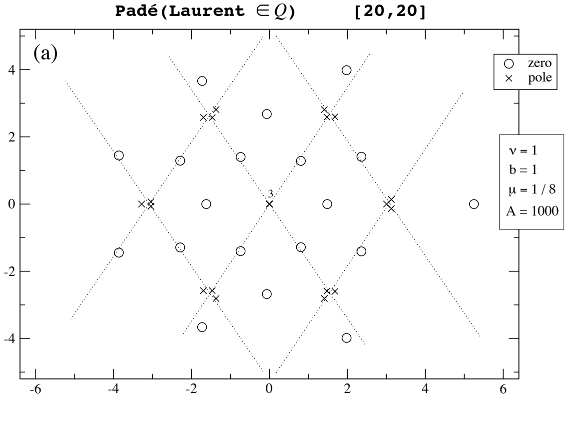


Figure 1. Plan complexe x du seul cas elliptique connu ($b^2 = 16\mu\nu$, $A =$ arb). La structure des zéros (symbole 0) et des pôles (symbole X) doit être doublement périodique, avec trois zéros simples et un pôle triple par parallélogramme période. Figure (d) : $(\nu, b, \mu, A) = (1, 16, 16, -52168)$, calcul de l’AP [30, 30] sur $\mathcal{Q}$. Les trois autres figures sont tracées pour $(\nu, b, \mu, A) = (1, 16, 16, -1408)$, donc $(g_2, g_3) = (64/3, 48)$. Figure (b) : emplacements exacts des zéros et des pôles, calculés à partir de la solution exacte (4.5). Figure (c) : zéros et pôles de l’AP [100, 100] calculé sur $\mathcal{R}$, montrant l’accumulation des doublets de Froissart sur le cercle unité (échelles différentes sur les deux axes). Figure (a) : zéros et pôles de l’AP [30, 30] calculé sur $\mathcal{Q}$. Dans tous les cas, on discerne nettement les trois zéros simples et le pôle triple de chaque période.

Remarque. La connaissance du nombre et des positions des zéros et des pôles d’une période permet la détermination numérique des invariants g_2 et g_3.



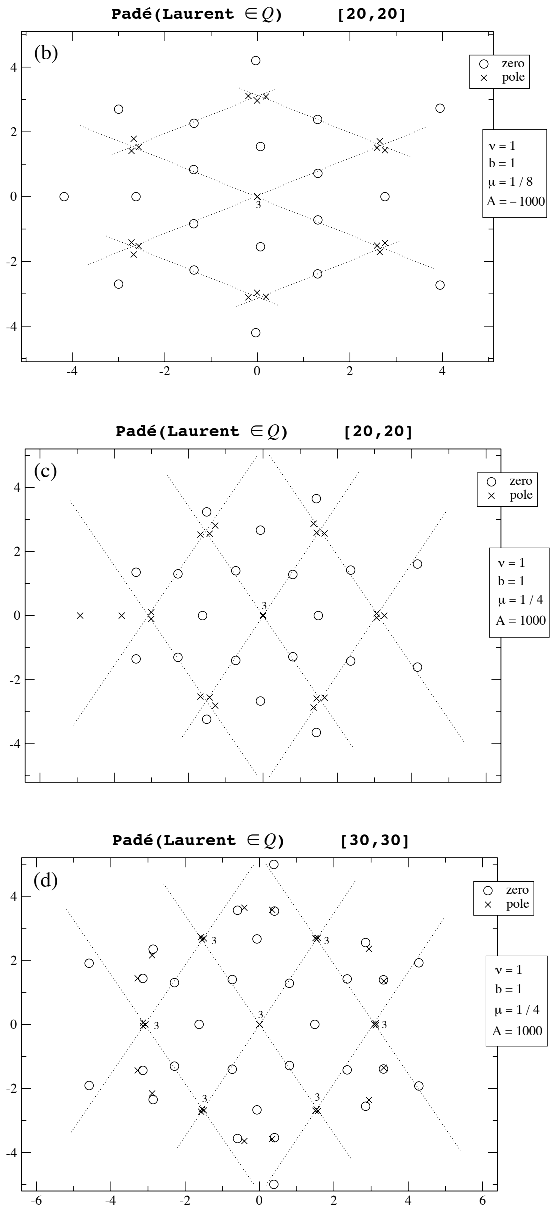


Figure 2. Plan complexe x pour les valeurs $b^2/(\mu\nu) = 4$ et 8 (cas génériques inconnus) et des AP calculés sur $\mathcal{Q}$. Fig. (a), $(\nu, b, \mu, A) = (1, 1, 1/8, 1000)$. Fig. (b), $(\nu, b, \mu, A) = (1, 1, 1/8, -1000)$. Figs. (c) et (d), $(\nu, b, \mu, A) = (1, 1, 1/4, 1000)$, pour $[L, M] = [20, 20]$ (Fig. (c)) et $[30, 30]$ (Fig. (d)). La structure est, avec une bonne précision, doublement périodique à trois zéros simples et un pôle triple par période.

6.2 Deuxième test d’ellipticité

Si la solution u est elliptique, alors, d’après un résultat classique, il existe deux fractions rationnelles R_1, R_2 telles que $u = R_1(\wp) + R_2(\wp)\,\wp'$, plus précisément

$$u(x, g_2, g_3) = \underbrace{\frac{\mathrm{Poly}_{N_1}(\wp)}{\mathrm{Poly}_D(\wp)}}_{\text{partie paire}} + \frac{\mathrm{Poly}_{N_2}(\wp)}{\mathrm{Poly}_D(\wp)} \sqrt{4\wp^3 - g_2\wp - g_3}, \tag{6.4}$$

où Poly_D désigne un polynôme de degré D.

En vue d’une recherche directe de la solution sous la forme (6.4), il est en principe possible de calculer numériquement les degrés N_1, N_2, D des fractions rationnelles de $\wp$, en remplaçant le plan complexe de x par celui de $\wp(x)$, de la manière suivante.

Tout d’abord, la nécessité d’une seule variable complexe dans les AP oblige à éliminer $\wp'$, ce qui introduit un branchement algébrique indésirable sous la forme $\sqrt{4\wp^3 - g_2\wp - g_3}$. Un moyen simple de le supprimer est alors, puisque $\wp$ est une fonction paire, de se restreindre à la partie paire de la série de Laurent $u(\chi = x - x_0)$,

$$u^{\mathrm{paire}} = -15b\chi^{-2} + \frac{b(56\mu\nu - 13b^2)}{19 \times 32\nu^2} - \frac{b\left(10\mu\nu - 3b^2\right)^2}{19^2 \times 64\nu^4}\chi^2 + \mathcal{O}(\chi^4), \tag{6.5}$$

d’inverser [14, vol. II, p. 527] la série de Laurent $\wp$ de x,

$$\wp(x, g_2, g_3) = x^{-2} + \frac{g_2}{20}x^2 + \frac{g_3}{28}x^4 + \mathcal{O}(x^6) \tag{6.6}$$

en la série de Laurent x^2 de $\wp$,

$$\begin{aligned} x^2 = \wp^{-1} \Bigg[& 1 + \frac{g_2}{20}\wp^{-2} + \frac{g_3}{28}\wp^{-3} + \frac{7g_2^2}{1200}\wp^{-4} + \frac{29g_2g_3}{3080}\wp^{-5} \\ & + \left(\frac{11g_2^3}{12480} + \frac{5g_3^2}{1274}\right)\wp^{-6} + \frac{167g_2^2g_3}{73920}\wp^{-7} \\ & + \left(\frac{77g_2^4}{509184} + \frac{669g_2g_3^2}{340340}\right)\wp^{-8} + \mathcal{O}(\wp^{-9}) \Bigg], \end{aligned} \tag{6.7}$$

puis de composer les deux séries (6.5) et (6.7) pour obtenir

$$\begin{aligned} u^{\mathrm{paire}} = & -15b\wp + \frac{b(56\mu\nu - 13b^2)}{19 \times 32\nu^2} + \left(\frac{3bg_2}{4} - \frac{b\left(10\mu\nu - 3b^2\right)^2}{19^2 \times 64\nu^4}\right)\wp^{-1} \\ & + \mathcal{O}(\wp^{-2}). \end{aligned} \tag{6.8}$$

La question est alors de tester si cette série (6.8), dont les coefficients dépendent de $(\nu, b, \mu, A, g_2, g_3)$, a bien pour somme une fraction rationnelle

$$u^{\mathrm{paire}}(x, g_2, g_3) = \frac{\mathrm{Poly}_{N_1}(\wp)}{\mathrm{Poly}_D(\wp)}. \tag{6.9}$$

La difficulté provient de la nécessité de donner, dans la série (6.8), des valeurs numériques, de préférence rationnelles, à (g_2, g_3) avant de pouvoir calculer les zéros et les pôles des AP. Puisque le parallélogramme période du plan x est déjà déterminé, il permet de calculer des valeurs approchées (complexes) de (g_2, g_3). Il importe donc d'étudier la sensibilité du résultat (valeurs de N_1, D) aux valeurs de (g_2, g_3), et cette sensibilité ne peut être testée que sur l'unique solution elliptique connue (4.5). La Figure 3 montre la structure des AP de (6.8) dans le plan complexe $\wp$ pour $b^2 = 16\mu\nu$ et pour un choix de (g_2, g_3) voisin de celui de ce cas.

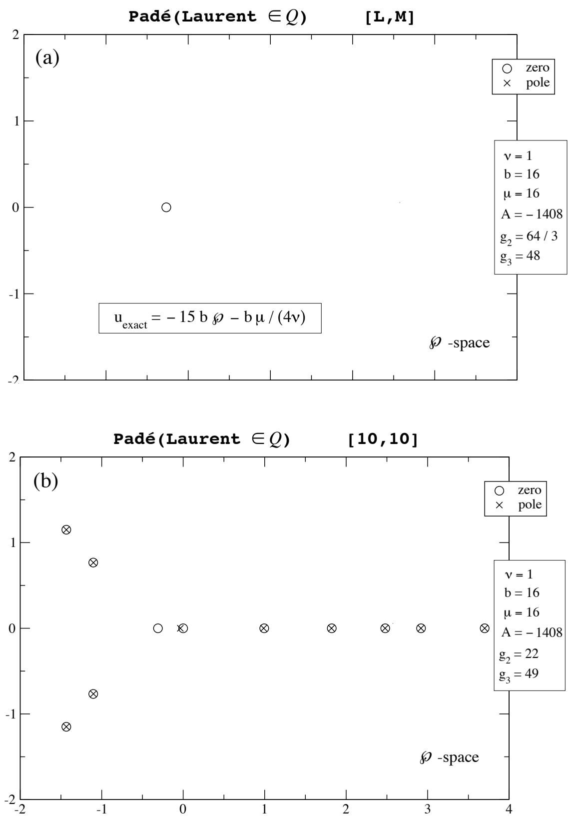


Figure 3. Plan complexe $\wp$ de la partie paire de la solution u pour $b^2/(\mu\nu) = 16$, $A = -1408$. La relation (4.5) définit les valeurs $g_2 = 64/3$, $g_3 = 48$. Dans la figure (a), $(\nu, b, \mu, A, g_2, g_3) = (1, 16, 16, -1408, 64/3, 48)$, tout AP $[L, M]$ est exact car calculé sur $\mathcal{Q}$, la figure contient donc l'unique zéro $\wp = -\mu/(60\nu)$ et elle n'a aucun pôle. Dans la figure (b), où les valeurs de (g_2, g_3) sont choisies rationnelles et légèrement différentes des valeurs exactes, $(\nu, b, \mu, A, g_2, g_3) = (1, 16, 16, -1408, 22, 49)$, l'AP $[10, 10]$, calculé sur $\mathcal{Q}$, fait constater 10 quasi-compensations d'un pôle par un zéro.

6.3 Détermination des degrés de la sous-équation

Suivant une suggestion de Belabas faite lors de la conférence, les degrés de la courbe algébrique (3.2) pourraient se calculer par une technique de Padé, comme suit. À partir de la série de Laurent (2.2), on construit d’abord une autre série de Laurent à pôle simple, par exemple $u'/u = \mathrm{Laurent}(x - x_0)$, on l’inverse en $x - x_0 = \mathrm{Laurent}(u'/u)$, et on la substitue dans la série $u = \mathrm{Laurent}(x - x_0)$. La série résultante $u = \mathrm{Laurent}(u'/u)$, qui réalise sous forme locale l’élimination de $x - x_0$ entre u et u', est alors analysée par Padé. Du branchement algébrique ainsi trouvé, il devrait être possible de déduire le degré en U de la courbe algébrique $G(U, V) = 0$, avec $U = u$, $V = u'/u$, donc les degrés de (3.2).

7 Approches analytiques

L’étude numérique précédente semble conclure à la nature elliptique de la solution cherchée. Nous présentons maintenant brièvement trois méthodes analytiques susceptibles de fournir les coefficients de la sous-équation (3.1). Si de plus celle-ci est supposée algébrique et à solution générale uniforme, alors elle a la forme nécessaire (3.2).

7.1 Équation de Monge–Ampère équivalente

La fonction inconnue de deux variables $F(u, u')$ qui définit la sous-équation d’ordre un (3.1) satisfait à une équation aux dérivées partielles très simple, obtenue par élimination de u'' et u''' entre (1.4) et les deux équations

$$\frac{\mathrm{d}}{\mathrm{d}x} F(u(x), u'(x)) = 0, \tag{7.1}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} F(u(x), u'(x)) = 0. \tag{7.2}$$

Dans la notation classique

$$\begin{aligned} &X = u, \quad Y = u', \quad P = \frac{\partial F}{\partial X}, \quad Q = \frac{\partial F}{\partial Y}, \\ &R = \frac{\partial^2 F}{\partial X^2}, \quad S = \frac{\partial^2 F}{\partial X \partial Y}, \quad T = \frac{\partial^2 F}{\partial Y^2}, \end{aligned} \tag{7.3}$$

c’est une équation de Monge–Ampère qui s’écrit

$$\nu \frac{Y^2(-Q^2 R + 2PQS - P^2 T) + Y P^2 Q}{Q^3} - bY \frac{P}{Q} + \mu Y + \frac{X^2}{2} + A = 0. \tag{7.4}$$

Notre problème équivaut alors à en trouver une solution particulière à zéro paramètre qui soit polynomiale. Ce nouveau problème pourrait n’être pas plus simple que l'original, car la méthode des caractéristiques est inapplicable puisque l'équation (7.4) hérite de la nature chaotique de KS. En particulier nous n'avons pas réussi à obtenir les degrés de cette solution polynomiale, si elle existe.

7.2 Construction de la sous-équation par l'algorithme LLL

Il existe une méthode [20, 13], mise au point en cryptographie, qui fournit *en nombres entiers* les coefficients de (3.2) pour m donné (et $a_{0,m}$ non normalisé à un). Connue sous le nom d'*algorithme de réduction du réseau* (lattice reduction algorithm) ou d'*algorithme LLL*, elle reçoit en entrée l'entier m, la suite des inconnues $a_{j,k}$ de (3.2), et L valeurs numériques (u_i, u'_i), $i = 1, \ldots, L$ (donc dans $\mathcal{R}$ et non pas dans $\mathcal{Q}$) de la somme des séries de Laurent u et u' aux points $x_i - x_0$, $i = 1, \ldots, L$. En sortie, si L est suffisamment grand devant le nombre $(m+1)^2$ d'inconnues $a_{j,k}$, elle exhibe en un temps polynomial une suite d'entiers $a_{j,k}$, par résolution du système linéaire homogène surdéterminé

$$\forall l = 1, \ldots, L : \quad \sum_{k=0}^{m} \sum_{j=0}^{2m-2k} a_{j,k} u_l^j u_l'^k = 0. \tag{7.5}$$

Si les entiers ainsi trouvés sont tous nuls, il faut augmenter m et recommencer. Nous n'avons pas encore mis cette méthode en œuvre car celle de la section suivante semble beaucoup plus prometteuse.

7.3 Construction de la sous-équation par la série de Laurent

Deux d'entre nous ont récemment proposé [23] une méthode analytique permettant de construire la sous-équation (3.2) à partir de la série de Laurent (2.2). L'algorithme est le suivant.

1. Choisir un entier positif m et définir l'EDO de Briot et Bouquet (3.2). Elle contient $(m+1)^2 - 1$ constantes inconnues $a_{j,k}$.

2. Calculer J termes de la série de Laurent, où J est légèrement supérieur à $(m+1)^2 - 1$,

$$u = \chi^p \left(\sum_{j=0}^{J} u_j \chi^j + \mathcal{O}\left(\chi^{J+1}\right) \right), \tag{7.6}$$

   avec dans le cas de KS $p = -3$.

3. Exiger que la série de Laurent satisfasse à l’EDO de Briot et Bouquet, c.à.d. demander l’annulation identique de la série de Laurent de $F(u, u')$ à l’ordre J

$$F \equiv \chi^D \left(\sum_{j=0}^{J} F_j \chi^j + \mathcal{O}(\chi^{J+1}) \right), \quad D = m(p-1), \ \forall j : \ F_j = 0. \qquad (7.7)$$

S’il n’existe pas de solution pour $a_{j,k}$, augmenter m et revenir à l’étape 1.

4. Pour toute solution, intégrer l’EDO autonome d’ordre un (3.2).

Le cœur de la méthode est la troisième étape, où le système de $J+1$ équations $F_j = 0$ aux $(m+1)^2 - 1$ inconnues $a_{j,k}$ est *linéaire* et *surdéterminé*, donc très facile à résoudre.

Quant à la quatrième étape, elle est résolue par des algorithmes de Poincaré, mis en œuvre par le progiciel *algcurves* [12].

Dans le cas de l’équation (1.4) étudiée ici, du fait de la valeur $p = -3$, la valeur minimale de m est $m = 3$, ce qui correspond à 15 inconnues $a_{j,k}$ et même à 10 inconnues seulement, à cause de la règle de sélection supplémentaire $m(p-1) \le jp + k(p-1)$, qui provient du degré de singularité $p = -3$ de u. Ces dix coefficients sont déterminés par le système de Cramer défini par les dix équations $F_j = 0$, $j = 0 : 6, 8, 9, 12$. Le système résiduel, qui est non-linéaire et surdéterminé aux inconnues fixes (ν, b, μ, A), a pour PGCD $b^2 - 16\mu\nu$ (cf. (4.5)), ce qui définit la solution de codimension un,

$$\begin{aligned} &\frac{b^2}{\mu\nu} = 16, \ \left(u' + \frac{b}{2\nu}u_s\right)^2 \left(u' - \frac{b}{4\nu}u_s\right) + \frac{9}{40\nu}\left(u_s^2 + \frac{60\mu^3}{\nu} + \frac{10A}{3}\right)^2 = 0, \\ &u_s = u + \frac{3b\mu}{2\nu}\cdot \end{aligned} \qquad (7.8)$$

Après division par ce PGCD, le système restant aux inconnues (ν, b, μ, A) admet exactement quatre solutions (il est suffisant d’arrêter la série à $J = 16$ pour obtenir ce résultat),

$$\begin{aligned} &b = 0, \\ &\left(u' + \frac{180\mu^2}{19^2\nu}\right)^2 \left(u' - \frac{360\mu^2}{19^2\nu}\right) + \frac{9}{40\nu}\left(u^2 + \frac{30\mu}{19}u' - \frac{30^2\mu^3}{19^2\nu}\right)^2 = 0, \end{aligned} \qquad (7.9)$$

$$b = 0, \quad u'^3 + \frac{9}{40\nu}\left(u^2 + \frac{30\mu}{19}u' + \frac{30^2\mu^3}{19^3\nu}\right)^2 = 0, \qquad (7.10)$$

$$\frac{b^2}{\mu\nu} = \frac{144}{47}, \quad u_s = u - \frac{5b^3}{144\nu^2}, \quad \left(u' + \frac{b}{4\nu}u_s\right)^3 + \frac{9}{40\nu}u_s^4 = 0, \qquad (7.11)$$

$$\frac{b^2}{\mu\nu} = \frac{256}{73}, \quad u_s = u - \frac{45b^3}{2048\nu^2},$$

$$\left(u' + \frac{b}{8\nu} u_s\right)^2 \left(u' + \frac{b}{2\nu} u_s\right) + \frac{9}{40\nu} \left(u_s^2 + \frac{5b^3}{1024\nu^2} u_s + \frac{5b^2}{128\nu} u'\right)^2 = 0. \quad (7.12)$$

À l'étape 4, on trouve alors la valeur un pour le genre de (7.8), et la valeur zéro pour celui de (7.9)–(7.12), puis les expressions (4.5) et (5.2) pour les solutions.

Avec la valeur minimale $m = 3$, on retrouve donc tous les résultats connus. Pour $m = 4$, il n'existe aucune autre solution. Le calcul pour $m \geq 5$ est actuellement en cours.

8 Conclusion

L'étude par approximants de Padé semble conforter la nature probablement elliptique de l'unique solution analytique de codimension nulle de l'EDO chaotique (1.4). Parmi les autres pistes, mi-analytiques, mi-numériques, qu'il serait possible de mettre en œuvre, il en existe une particulièrement intéressante car constructive, c'est celle d'obtention de la sous-équation par la série de Laurent. Son seul inconvénient est l'absence de borne supérieure pour le degré m, ce qui contraint à de volumineux calculs qui sont encore en cours.

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Hybrid Systems

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Abstract. The hybrid systems' field was born from problems common to two disciplines: automatic and computer science. In this article, we present a mathematical formalization of hybrid systems, describing their definition and their behavior. We illustrate the concept by two examples. We introduce a new method of integration, adapted for problems issued from hybrid systems' modeling and we apply this method to a classical example.

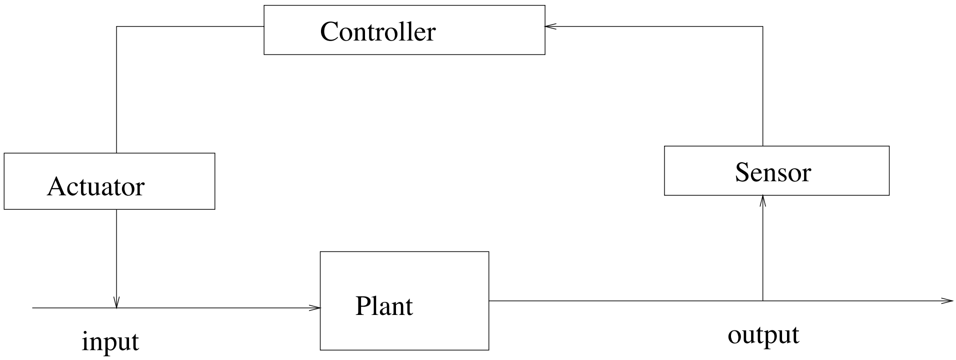
1 The hybrid automaton model

1.1 Motivation for hybrid system modeling

Perhaps the simplest way to understand the necessity of introducing the concept of hybrid systems is to look at a physical system from a control point of view. The basic paradigm in control theory is based on the following schema:

The mythical plant is the considered system: it is a part of a car, a chemical reactor, a cell … This plant evolves during time under the supervision of a controller. This controller could be another system, but in the hybrid systems framework it can be thought as a program (built in some microprocessor). This program is a digital or discrete device. Theoretically it is represented as a finite automata (it is the discrete part of the hybrid system). Under the output of the program the physical (or analog) device will jump from its actual state to another legal state.

Two other basic devices must be added to the previous diagram: sensors and actuators. The sensors (or A/D devices) are physical objects which are able to make observations on the system (measures) and to send these results as input to the program of the controller. For this they must be able to translate physical (analog) measures



into digital quantities. From a practical point of view, one of the problem is that these measures cannot be done continuously but at discrete time. This is a dramatic constraint because the system evolves between two successive measures. We will consider this problem in the text.

Having as inputs the digital values of the sensors, the program will compute what has to be applied to the plant in order to achieve a desired objective.

Of course the program must be built for the achievement of such an objective (it is the problem known as controller synthesis). After the program computation, its outputs will be transmitted to the plant through a device which turned digital output into analog input for the plant. These devices are the actuators (or D/A devices).

What we hope is that the plant will follow the desired schedule and will not fall into undesirable behaviors.

Now our objective is to put this informal description into a *mathematical* model.

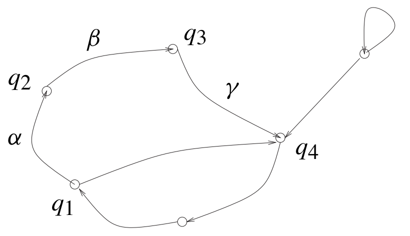
1.2 From transition systems to hybrid systems

1.2.1 Transition systems. Let us start by the definition of a basic structure in computer science.

Definition 1.1. A transition system is a collection $\mathcal{T} = \{Q, \Sigma, \rightarrow, Q_i, Q_f\}$ where:

- Q is the set of states of $\mathcal{T}$.
- Σ is the alphabet of events.
- $\rightarrow$ is a subset of $Q \times \Sigma \times Q$. It is the subset of legal transitions.
- $Q_i \subset Q$ represents the subset of initial states.
- $Q_f \subset Q$ represents the subset of final states.

In our study Q and Σ will be finite sets. One can picture a transition system by a graph whose nodes are the states and whose edges are labeled by the alphabet's letters.



For example, in the above figure we have represented the transition $q_1 \alpha q_2$. If there is no ambiguity, we can speak of the α transition.

Some remarks are useful:

• A *state* is a formal object, but from our point of view it has to be imagined as an equilibrium point of a dynamical system.

• There is no notion of *time* in a transition system. The system can stay in a given state as long as an event (labeled by an element of Σ) does not occur. But if a transition take place, this happens with a null duration. This is a deep and subtle problem in computer science.

• The previous schema shows the possibility of *indeterminism* in the behavior of the system.

A. Behavior of a transition system. A set of words (or *traces*) Σ^* is associated to the alphabet Σ. It is the set of all finite sequences of elements of Σ and the empty word $\emptyset$. Σ^* is a monoid provided with the concatenation intern law.

The action of the monoid Σ^* on Q

$$\Phi : Q \times \Sigma^* \to Q$$

is defined by:

1. $\Phi(q, \emptyset) = q, \quad \forall q \in Q.$
2. $\Phi(q, \sigma \times \sigma') = \Phi(\Phi(q, \sigma), \sigma'), \quad \forall \sigma, \sigma' \in \Sigma^*$ and $q \in Q$.

The second part of the definition has a meaning only if the two members of the equality are defined. The transition system *accepts* a word σ if and only if there exist $q \in Q_i$ and $q' \in Q_f$ such that $\Phi(q, \sigma) = q'$. For each subset $\mathcal{Q} \subset \Sigma^*$, we denote by $\mathcal{L}(Q)$ the set of all the executable words and we have the following remark:

Remark 1.2. If Q_f is empty, then $\mathcal{L}(Q)$ is the subset of Σ^* of states that can be reached by the flow associated to the system (in general there exits unreachable states) with the restriction that we must start in Q_i.

The *reachability problem* for a transition system is the problem of proving the non vacuity of $\mathcal{L}(Q)$ in view of the couple (Q_i, Q_f).

B. The problem of time. The success of automata theory (and of transition systems) in computer science is related to the fact that it can be used to model the logical behavior of a program, as well as for the specification and verification of systems. But it becomes clear that this formalism has some severe drawbacks when the goal is to model physical systems. The following quotation is extracted from Rajeev Alur and David L. Dill [1]:

> Although the decision to abstract away from quantitative time has had many advantages, it is ultimately counterproductive when reasoning about systems that must interact with physical processes; the correct functioning of the control system of airplanes and toaster depends crucially upon *real-time* considerations. We would like to be able to specify and verify models of real-time system as easily as qualitative models. Our goal is to modify finite automata for this task and develop a theory of *timed* finite automata, similar in spirit to the theory of ω-regular languages. We believe that this should be the first step in building theories for real-time verification problem.

Now the goal is to handle time in a satisfactory way. The basic idea of [1] is to introduce *clocks* and *clock constraints*.

Definition 1.3. A clock is an application c from $\mathbb{R}_+$ to $\mathbb{R}_+$. A clocks system is a finite family of clocks $\{c_i\}_{i=1,\dots,n}$.

When we start the study of the system we will always suppose that all the clocks have the value zero.
Now, let $X = \{x_1, \dots, x_m\}$ be a finite set and $\Lambda(X)$ a family of constraints built on X defined inductively by:

- the variable x_i;
- elementary constraints of two forms: $A := x_i \leq \alpha$ or $B := \alpha \leq x_i$ $(\alpha \in \mathbb{Q})$;
- if $B \in \Lambda(X)$, then $\neg B \in \Lambda(X)$;
- if $B, C \in \Lambda(X)$, then $B \wedge C \in \Lambda(X)$.

Definition 1.4. An element a of $\mathbb{R}^m$ satisfies $\Lambda(X)$ if each δ belonging to $\Lambda(X)$ is evaluated to true using the value a.

Then we can associate to $\Lambda(X)$ the following set:

$$\mathcal{I}(X) = \{a \in \mathbb{R}^m; \quad a \text{ satisfies } \Lambda(X)\}.$$

As $\mathcal{I}(X)$ is a subset of $\mathbb{R}^m$, we can speak of the boundary $\partial(\mathcal{I}(X))$ of that region.

1.2.2 Timed automata. A *timed automata* will be built of several components:

1. a transition system $\mathcal{T} = \{Q, \Sigma, \rightarrow, Q_i, Q_f\}$,
2. a set $\mathcal{C} = \{c_i\}_{i=1,\ldots,m}$ of clocks,
3. a set $\Lambda_{\mathcal{C}}(q) = \Lambda(q)$ of constraints built on $\mathcal{C}$ for each state q,

and of several applications:

- $$\begin{aligned} \widetilde{c} : \mathbb{R}_+ &\longrightarrow \mathbb{R}_+^m \\ t &\longrightarrow (c_1(t), c_2(t), \ldots, c_m(t)) \end{aligned}$$

- $$\widetilde{q} : \mathbb{R}_+ \longrightarrow Q$$

The first application specifies the clocks values at time t, the second one specifies the system state at time t (it is a piecewise constant map).

We also need a convention: if the system will undergo a transition from a state q to a state q' at a time t, then we will denote by t^- (respectively t^+) the value of t before the transition (respectively after).

The model needs to verify the following two assumptions:

P$_1$ *Every transition is done in zero time.* We can write $t = t^+ = t^-$.

P$_2$ *At each time t the system can be observed and it can be decided if $\widetilde{c}(t) \in \mathcal{I}(\mathcal{C})$.*

The second property says that we are able to compute the value $\widetilde{c}(t)$ and we can decide automatically (by a zero-time algorithm) if $\widetilde{c}(t)$ verify the set $\Lambda(q)$ of constraints. These two assumptions will have important consequences on the system behavior.

These conventions are useful for describing the behavior of the automata: during the evolution it can happen that the point $\widetilde{c}(t)$ will meet the boundary ∂_q of $\mathcal{I}(\Lambda(q))$. Then the system needs to be reset and the following operations must occur:

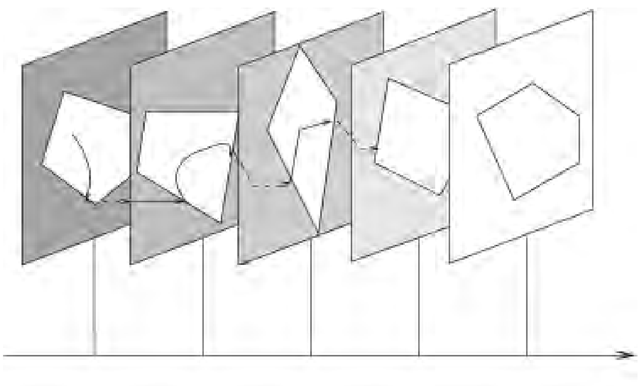
- At t^- we know (following P$_2$) that we are in ∂_q, in the state q. The assumption is that we need to jump to another state. For this, we look at the possible transitions described by the transition system.
- In the list of possible transitions of the form $q \rightarrow *$ allowed by $\mathcal{T}$ we have to select one of them. Suppose that it is $q\sigma q'$.
- Then a new map

  $$\begin{aligned} \mathrm{Reset}_{q\sigma q'} : \partial_q &\longrightarrow \mathcal{I}(\Lambda(q')) \\ \widetilde{c}(t^-) &\longrightarrow \widetilde{c}(t^+) \end{aligned}$$

  will describe the new starting point of the system.

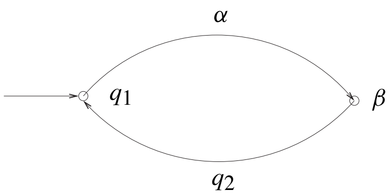
Behavior of a timed automata. Now the behavior can be described in the following informal way:

- By definition the system starts in a state q^0 belonging to Q_i. Using the previous notations we have $\widetilde{q}(0) = q^0$ and $\widetilde{c}(0) = 0$.
- We must verify that $\widetilde{c}(0) = 0$ belongs to $\mathcal{I}(\Lambda(q))$, if not the system must be stopped.
- If $\widetilde{c}(0) = 0$ belongs to $\mathcal{I}(\Lambda(q))$, then the system stays in the q^0 state as long as $\widetilde{c}(t)$ belongs to $\mathcal{I}(\Lambda(q))$. If this is the case, then the experience is done.
- If not, there is a first time $t_1^- = t_1$ such that $\widetilde{c}(t_1) \in \partial_q$. Now a transition must occur: there exist at least a $\widetilde{q}(t_1^+) \in Q$ and an $\alpha \in \Sigma^*$ such that the system will be submitted to the transition $\widetilde{q}(t_1^-)\alpha\widetilde{q}(t_1^+)$ and the clocks values must be reset.
- This operation is described by the reset function $\mathrm{Reset}_{\widetilde{q}(t_1^-)\alpha\widetilde{q}(t_1^+)}$ and we obtain a new starting point in $\mathcal{I}(\Lambda(q))$. By our notation this new point is $\widetilde{c}(t_1^+)$.
- Then we repeat the discussion starting at time $t_1^+ = t_1$.



Example 1.5. Using the previous notations, consider:

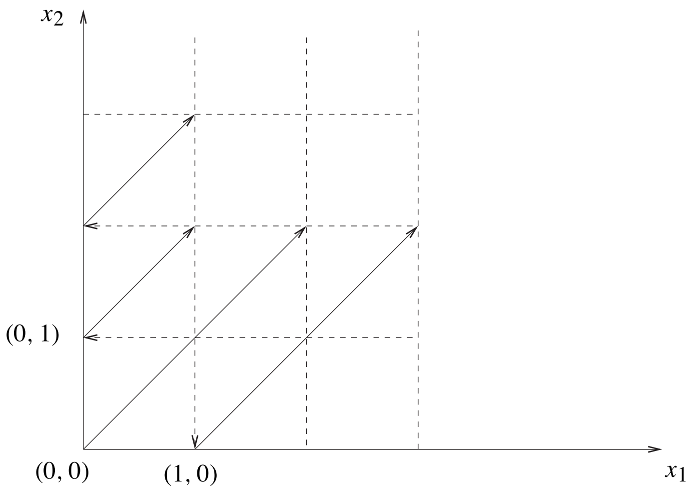
- $Q = \{q_1, q_2\}$, $\Sigma = \{\alpha, \beta\}$.
- The graph of $\mathcal{T}$ is:



- We suppose that $Q_i = \{q_1\}$ and $Q_f = \{q_1\}$.
- $\mathcal{C} = \{c_1, c_2\}$ and $\Lambda(q_1) = \{x_1 \leq 3 \wedge x_2 \leq 2\}$, $\Lambda(q_2) = \{x_1 \leq 1\}$. $\mathcal{I}(\Lambda(q_1)) = \{(x_1, x_2) \in \mathbb{R}^2,\ 0 \leq x_1 \leq 3 \text{ and } 0 \leq x_2 \leq 2\}$, $\mathcal{I}(\Lambda(q_2)) = \{(x_1, x_2) \in \mathbb{R}^2,\ 0 \leq x_1 \leq 1\}$.
- $\mathrm{Reset}_{q_1 \alpha q_2}(x_1, x_2) = \{(0, x_2)\}$ and $\mathrm{Reset}_{q_2 \beta q_1}(x_1, x_2) = \{(x_1, 0)\}$.

Now we must give the clocks expression. We suppose in this example that $c_1(t) = t$ and $c_2(t) = t$.

Now the presentation is complete and it is useful to represent the timed behavior of the automata by a trajectory in the *time space* of the clocks.



There is a huge number of interesting questions associated to the concept of timed automata. Nevertheless, the theory is growing very fast. But in our presentation we will introduce the next step in the modelisation process.

Description of the experimental protocol. We know the dynamic behavior of a system (i.e. its evolution during a given time interval) through a finite number of measurements.

We are thus naturally led to introduce variables x_i representing the information that one can obtain from the system and which are accessible to measurement (for example the temperature, the velocity, the acceleration …). A reasonable assumption is made that there is a finite number of these variables.

We will observe the evolution of the running process in a space $\mathbb{R}^n$ by measuring the value of these variables.

The assumption which is made is that this evolution is *locally* described by a dynamic system. This means that it is supposed that the variation law of the x_i's is valid only locally.

On the other hand it is reasonable to say that the model limits are accessible to measurement. If the variables reach certain thresholds, then it will be necessary to change the model.

We then naturally propose "trajectories" which are not differentiable in general. However, as the physical evolution of the variables is continuous, it will be reasonable to suppose the trajectories continuous and piecewise derivable (the natural framework will be thus that of the absolutely continuous functions).

1.3 Measurement

The decisions on the system (i.e. of the actions on this system thanks to actuators) will be made by comparing these measurements (provided by sensors) with references.

These references will be defined by a closed family in the boolean sense of constraints.

A measurement is made on certain quantities (in a finite number) in a given time. If we suppose that we will always have the same number n of variables x_i, then it is an application

$$m : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}.$$

It is however inaccurate to suppose that these measurements are accessible in any place and at any moment. A sensor cannot function in real time.

This remark can be compared with that of R. Thom [8, pp. 35]:

"Ici se pose un problème préjudiciel sur lequel, même actuellement, les physiciens ont des opinions divergentes; les fonctions mesurées $g(x, t)$ sont définies sur un réseau fini de points, car on ne peut introduire dans B qu'un nombre fini de sondes et faire un nombre fini de mesures (la vie humaine est finie et chaque mesure exige un temps fini pour être menée à bien). Quelles hypothèses faut-il faire sur la nature mathématique de ces fonctions qui ne sont connues que sur un ensemble fini de points de leurs graphes? Sont-elles analytiques (voir même des polynômes) ou au contraire simplement différentiables ? La réponse est, je crois, très simple: tout dépend de ce que l'on veut faire. Si l'on veut éprouver un modèle quantitatif précis, il faut faire des calculs et des vérifications numériques: or, seules les fonctions analytiques (en fait les polynômes de bas degré) sont calculables; en ce cas, les fonctions $g(x, t)$ seront analytiques et de préférence, données par des formules explicites. Si l'on ne s'intéresse seulement au point de vue de la stabilité structurelle du processus, il est au contraire beaucoup plus naturel de supposer les fonctions $g(x, t)$ seulement m fois continûment différentiables (m petit, deux ou trois dans la pratique). Ainsi l'état local ou global de notre système sera paramètré par un point d'un espace fonctionnel, variété banachique, situation analogue à celle définie par le procédé de préparation." [1]

The abstract description of the mathematical model with respect to the process evolution is the following:

1. Suppose we know a value $X(0) = (x_1(0), \ldots, x_n(0))$ of the variables associated to the process at the moment $t = 0$. Moreover, suppose these variables to be real-valued.

2. We make the assumption that the variables evolution can be *locally* represented by a dynamic system of the form:
$$X' = f(X). \tag{1.1}$$

3. The system will be observed starting from $t = 0$.

4. We suppose that these values are used through measurements h_i, $i = 1, \ldots, M$. Thus at certain moments t we suppose that we know how to calculate $h_i(X(t), t)$ and how to compare this number with 0 (we can always arrive to this situation).

5. It is supposed that as long as
$$h_i(X(t), t) \leq 0, \quad \forall i,$$
the system evolution is representable by (2.1).

6. If there exists an index i at the t moment such that $h_i(X(t), t) > 0$, then the model is not valid anymore (it is the basic assumption) and we must consider a new dynamic system and possibly new functions m_i.

7. We make also the assumption that there is only a finite number of possible systems. We call a "mode" the set of the acceptable field and the associated differential system.

Now let us represent into mathematical terms the former abstract description.

[1] "Here there is a difficulty on which, even currently, the physicists have divergent opinions; the measured functions $g(x, t)$ are defined on a finite points network, because one can introduce into B only a finite number of probes and make a finite number of measurements (the human life is finite and each measurement requires a finite time to be concluded). Which assumptions are necessary to make on the mathematical nature of these functions which are known only on a finite set of their graph points? Are they analytical (see even polynomials) or on the contrary only differentiable? The answer is, I believe, very simple: all depends on what one wants to do. If one wants to test a precise quantitative model, it is necessary to make numerical calculations and checks: however, only the analytical functions (in fact the polynomials of low degree) are calculable; in this case, the functions $g(x, t)$ will be analytical and, will preferably be given by explicit formulas. If one is interested only from the point of view of the structural stability of the processus, it is on the contrary more natural to suppose the functions $g(x, t)$ only m times continuously differentiable (m small, equal to two or three in practice). Thus the local or global state of our system will be parameterized by a point of a functional space, Banach variety, situation similar to that defined by the preparation method."

1.4 Hybrid systems

1. **The state space of the hybrid system.**

   - We denote by $\mathcal{Q} = \{q_1, \dots, q_m\}$ the modes set of the system. It is a finite set.[2]
   - The state space of the system will be $\mathbb{R}^n$. (Naturally it would be possible to have for each mode a state space of different size for each mode). X will represent a point of $\mathbb{R}^n$ and $X(t)$ will be the system state at the moment t in a certain mode.
   - Each mode will be associated to a dynamic system on $\mathbb{R}^n$ of the form:

   $$\frac{dX}{dt}(t) = f_q(X(t)), \tag{1.2}$$

   with $q \in \mathcal{Q}$ and $t \geq 0$. We suppose that f_q is of class $\mathcal{C}^1$.

   At a given moment the system state is described by a couple $(X(t), q)$ belonging to $\mathcal{S} = \mathbb{R}^n \times \mathcal{Q}$. In order to be more precise we must denote by Ω_q the definition field of f_q and introduce $\Omega = \bigcup_{q \in \mathcal{Q}} \Omega_q$.

   - The system evolution will be then described by two applications:

   $$\widetilde{X} : \mathbb{R}^+ \to \mathbb{R}^n$$

   $$\widetilde{q} : \mathbb{R}^+ \to \mathcal{Q}.$$

   $\widetilde{X}$ will be a piecewise differentiable function and $\widetilde{q}$ will be piecewise constant. We normalize these functions by supposing they are right-continuous and have left-hand limits. The temporized state of the system is the triple:

   $$S(t) = (\widetilde{X}(t), \widetilde{q}(t), t).$$

2. **The transition structures of the system.**
   This structure is associated to:

   - A finite set of functions:

   $$h_i : \mathbb{R}^n \times \mathbb{R}^+ \times \mathcal{Q} \to \mathbb{R}$$

   with $i \in I$ a finite set of indices, describing the fields $\Omega_q \subset \mathbb{R}^n$: $\Omega_q = \{x \in \mathbb{R}^n, \;\; h_i(x, t, q) \leq 0\}$.

   - A real $\epsilon > 0$.

[2] We changed the name of state into that of mode because we want to preserve the concept of state to describe the state $X(t)$ of the system at the moment t in accordance with the uses of the dynamic system theory.

- These two data allow us to build an alarm set Σ_ε (as soon as we enter this set, the system must do a transition) as follows:

$$\Sigma_{i,\epsilon} = \Sigma^+_{i,\epsilon} \cup \Sigma^-_{i,\epsilon}, \quad i \in I$$

with

$$\Sigma^+_{i,\epsilon} = \{(x,t,q), h_i(x,t,q) \leq \epsilon\}$$

and

$$\Sigma^-_{i,\epsilon} = \{(x,t,q), h_i(x,t,q) \geq -\epsilon\}.$$

The set

$$\Sigma_\epsilon = \bigcup_{i \in I} \Sigma_{i,\epsilon}$$

will be the *alarm* set of the system.

- We will suppose that there is a multiapplication of transition:

$$\mathrm{T} : \Sigma \rightsquigarrow \{\Omega \times \mathbb{R}^+ \times \mathcal{Q}\} \setminus \Sigma$$

such that, if $(y,s,p) \in \mathrm{T}(x,t,q)$, then we have:

  - $s \geq t$,
  - $y \in \Omega_p$.

The precise definition of this application is the operation key of the hybrid system.

1.4.1 System trajectory. We must introduce:

- A new function Θ defined on $\{\Omega \setminus \Sigma\} \times \mathbb{R}^+ \times \mathcal{Q}$ by

$$\Theta(x,t,q) = \inf\{s > t, \Phi_q(s,x,t) \in \Sigma\}.$$

It is the instant when the system trajectory $X' = f_q(X)$ starting from the point (x,t) reaches for the first time the alarm set. We denote by $\Phi_q(s,x,t)$ the value at the moment s (if it exists) of the flow associated with the fields f_q starting from x at the moment t (naturally $s \geq t$). If $\Theta(x,t,q) = +\infty$, then the system will never intersect the alarm set.

- A real number $\Delta > 0$ (which is the necessary time for evaluation and calculus).

- A standby time $T > 0$. (This standby time can be introduced as constraints h_i, we do not do it for the sake of clearness).

- The variable (x,t,q).

A trajectory will be defined by the following pseudo-algorithm:

1. We initialize $(x, t, q) \leftarrow (x_0, 0, q_0)$ (the initial condition). **Output:**

$$\begin{cases} \widetilde{X}(0) := x_0 \\ \widetilde{q}(0) := q_0 \end{cases}$$

2. **While** $(x, t, q) \in \Sigma$ **and** $t \leq T$:

$$\begin{cases} (x, t, q) \overset{\sim}{\leftarrow} \mathrm{T}(x, t, q) \\ (x, t, q) \leftarrow (x, t + \Delta, q) \end{cases}$$

Output:

$$\begin{cases} \widetilde{X}(t) := x \\ \widetilde{q}(t) := q \end{cases}$$

3. **While** $(x, t, q) \notin \Sigma$ **and** $t \leq T$:
If $(x, t, q) \notin \Sigma$, **then**

$$\{ \; (x, t, q) \leftarrow \Phi_q(t + \Delta, x, t)$$

Output:

$$\begin{cases} \widetilde{X}(t) := x \\ \widetilde{q}(t) := q \end{cases}$$

4. **If** $t \geq T$ then **stop**.

Remark 1.6. The previous method is not an algorithm for several reasons:

- The step $(x, t, q) \overset{\sim}{\leftarrow} \mathrm{T}(x, t, q)$. This means that we have to select an element in the set $\mathrm{T}(x, t, q)$, image of (x, t, q) by the multiapplication T. The method of choosing this selection will be studied later.

- The second point which must be clarified is that of the predicate evaluation $(x, t, q) \in \Sigma$. It is a decision problem depending on the data of Σ and on the vectors fields.

We denote $t_n = n\Delta$. The system *trajectory* could be defined in two different ways:

- **Discrete trajectory**
We defined in the algorithm the sequence $(\widetilde{X}(t_n), \widetilde{q}(t_n))$. It is a discrete trajectory.

- **Continuous trajectory**
We can rebuild the trajectory on $[0, T]$ by considering on $[t_n, t_{n+1}]$ (where we have the knowledge of (x, t, q)) the function $\widetilde{q}(t)) = q$ and $\widetilde{X}(t)) = \Phi_q(t - t_n, x, t_n)$.

1.4.2 Construction of the transition multiapplication. We have to give a description of $\mathrm{T}(x, t, q)$ when (x, t, q) belongs to Σ_ϵ.

- We associate to (x, t, q) a subset $\mathcal{I}((x, t, q)) \subset I$ formed of indices i such that $(x, t, q) \in \Sigma_{i,\epsilon}$. These are the active constraints at (x, t, q).
- We have to supposed the existence of a functions family $g_{i,j}$ defined on $\Omega \times \mathcal{Q} \times \mathbb{R}^+$ by:

  $$g_{i,j}(x, t, q) = (y, s, p)$$

  with

  - $s \geq t$,
  - $i = q$,
  - $j = p$.

  The significance of this application is the following: when we are in an "observed" state, a certain number of actions are possible on the system. These possibilities depend on the actuator. It does not mean that the system must effectuate these actions.

  The support $G_{i,j}$ of $g_{i,j}$ is the set of points where $g_{i,j}$ is defined. These supports can intersect each other.
- Thus if the system is in the state (x, t, q), the following actions happen:
  1. The sensors verify if $(x, q, t) \in \Sigma$.
  2. **If not:** The system either awaits the end of the observation time in order to restart the state analysis, or it must ask an objective function (we will not analyze this case for the moment).
  3. **If yes:** The system considers the possibilities defined by the functions $g_{i,j}$ and by the active constraints $\mathcal{I}((x, t, q))$. A decision must be taken according to a strategy defined beforehand.
  4. In the case of a defined strategy, we form $g_{q,p}(x, t, q) = (y, t + \Delta, p)$ (by supposing that Δ is uniform).

Remark 1.7. We can interpret *in a confined case* the model in another manner. We can introduce a family $f(X, q) = f_q(X)$ of vectors fields on $\mathbb{R}^n$, indexed by $\mathcal{Q}$. In $\mathbb{R}^n$ we can also introduce a domains family Ω_q and then form the vectors field

$$F(X) = \sum_{q \in \mathcal{Q}} f_q(X) X_{\Omega_q}(X),$$

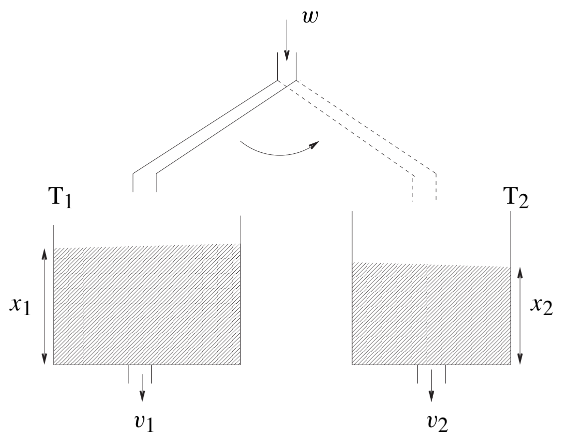
where X_{Ω_q} indicates the characteristic function of Ω_q. We thus see an interpretation close to that of the control theory.

However, by doing this, we impose that Ω_q are disjoined. But our definition *does not impose* this constraint. If (x, q, t) is not in Σ, we must admit that x can belong to several domains Ω_p, Ω_q being one of them. It is a source of indeterminism for the system but it is also a richness of the model.

2 Some examples of hybrid automaton [6]

2.1 The system of two tanks

There are two tanks of water, T_1 and T_2, each one containing liquid draining from its bottom with a fixed flow v_i, $i = 1, 2$ (the flow from each tank may be different). There is an overhead spout that can be positioned above only one tank at a given time and pours liquid into this one at same constant rate w. Denote by x_i the water volume in the tank i, $i = 1, 2$.



The goal is to maintain the fluid level of each tank above zero. In order to do this, we must choose a switched control strategy that changes the inflow into the tank 1 whenever $x_1 \leq 0$, respectively into the tank 2 whenever $x_2 \leq 0$.

The change in each water level tank is a continuous-time function that may be modeled by ordinary differential equations. On the other hand, the movement of the server spout is a function that can take only two discrete values, i.e. the spout is either above the first or above the second tank.

We have two discrete states for the system:

- The state q_1 corresponds to the fulfilling of the first tank (and then the second one is not fulfilled).

- The state q_2 corresponds to the fulfilling of the second tank (and then the first one is not fulfilled).

The two legal transitions correspond to switches from state $q_1 \to q_2$, respectively $q_2 \to q_1$.

$$\begin{tikzcd} q_1 \arrow[rr, bend left, "(q_1, q_2)"] & & q_2 \arrow[ll, bend left, "(q_2, q_1)"] \end{tikzcd}$$

Physically we suppose that the tanks have a unit section such that the volume of liquid in a tank is measured by its height. We suppose also $0 \le x_i \le H_i$, where $H_i > 0$ is the total volume of the tank T_i.

To each node a domain is associated (called invariant), written as:

$$\Omega(q_1) = \Omega(q_2) = \{(x_1, x_2);\ 0 < x_1 < H_1,\ 0 < x_2 < H_2\}.$$

In the mode $(q_1, \Omega(q_1))$, the dynamic is described by the following differential system:

$$\begin{cases} \dot{x_1} = w - v_1 \\ \dot{x_2} = -v_2 \end{cases} \quad \text{for } (x_1, x_2) \in \Omega(q_1).$$

In the second mode $(q_2, \Omega(q_2))$ we have:

$$\begin{cases} \dot{x_1} = -v_1 \\ \dot{x_2} = w - v_2 \end{cases} \quad \text{for } (x_1, x_2) \in \Omega(q_1).$$

We have to define the interface between two modes. Let us suppose that we are in the mode q_1. We can switch in two situations:

- either when $x_1 \ge H_1$: whatever the situation in the tank T_2 may be, we must switch in order to avoid the overflow in the tank T_1;

- or when $x_2 \le 0$: whatever the situation in tank T_1 may be, we must switch in order to avoid the emptiness of the tank T_2.

The guards are then described by the following sets:

$$G(q_1 \to q_2) = \{(x_1, x_2) \in \mathbb{R}^2;\ x_1 \ge H_1 \text{ or } x_2 \le 0\},$$

$$G(q_2 \to q_1) = \{(x_1, x_2) \in \mathbb{R}^2;\ x_2 \ge H_2 \text{ or } x_1 \le 0\}.$$

The Reset functions are:

$$\text{Reset}_{(q_1 \to q_2)}((x_1, x_2)) = ((x_1, x_2)),$$

$$\text{Reset}_{(q_2 \to q_1)}((x_1, x_2)) = ((x_1, x_2)).$$

The behavior of the system is the following:

- We start at $(x_1^0, x_2^0) \in \text{Init} = \{X \in \mathbb{R}^2; 0 \le x_1 \le H_1 \text{ and } 0 \le x_2 \le H_2\}$.
- Without loss of generality, (x_1^0, x_2^0) can be supposed to belong to T_1 (the incoming flow is in this tank).

Two situations are possible: either the second tank T_2 becomes empty in a certain time and then we will switch w to T_2 (Figure 1), or the first tank T_1 will be fulfilled and then w will be changed into T_2 (Figure 2).

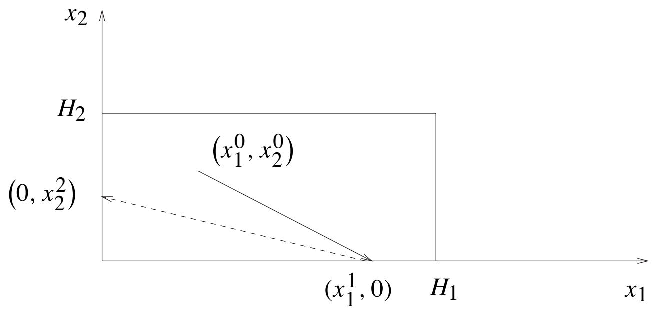


Figure 1

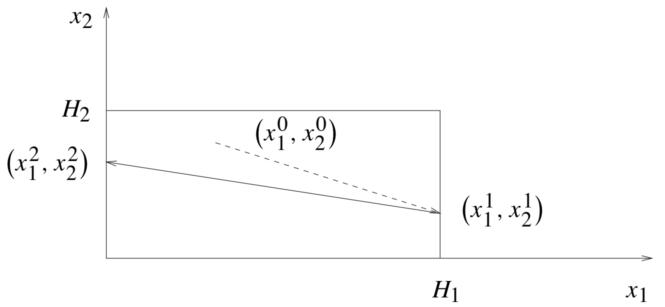


Figure 2

- We use then the differential system corresponding to the state q_1 and we suppose that at the time $t_0 = 0$, $x_i = x_i^0$, $i = 1, 2$. For simplicity we chose $x_1^0 = 0$ and $x_2^0 = h < H_2$.

In fact the dynamic is under the control of three constants, v_1, v_2 and w and of the upper-levels H_1 and H_2. We could remark that $\dot{x_1} + \dot{x_2} = w - v_1 - v_2 = a$, a being a constant. It means that

$$x_1 + x_2 = at + b,$$

where b is an integration constant, given by the initial condition. By the above supposition, we have $b = h$.

We have then three possibilities:

1. $\boldsymbol{a < 0}$**.** Both tanks will be empty in a time $t^* = -\frac{h}{a}$.
2. $\boldsymbol{a = 0}$**.** The global amount of liquid is being conserved and we have a periodical solution.
3. $\boldsymbol{a > 0}$**.** Both tanks will be overflowed in a time $t^{**} = \frac{H_1+H_2-h}{a}$.

Execution. The system starts at the time $t_0 = 0$ in the mode q_1. The initial point is $x_1^0 = 0$, $x_2^0 = h$ verifying the condition $0 < h < H_2$. The solution is:

$$\begin{cases} x_1(t) = (w - v_1)t \\ x_2(t) = -v_2 t + h \end{cases}$$

The system will meet a guard at the moment t_0^+ and then will quit the mode q_1. There are two possibilities:

- either $x_2 = 0 \Longleftrightarrow t_0^+ = \frac{h}{v_2}$,
- or $x_1 = H_1 \Longleftrightarrow t_0^+ = \frac{H_1}{w-v_1}$.

We restart the same method in the mode q_2. The important thing is that we obtain an infinite sequence of switches between the modes. We calculate explicitly the sum

$$\tau_\infty = (t_0^+ - t_0) + \sum_{i=0}^{+\infty}(t_{i+1}^+ - t_i^+)$$

and we obtain $\tau_\infty = -\frac{h}{a}$.

We prove then the following result:

- if $w < v_1 + v_2$ $(a < 0)$
- or if $w > v_1 + v_2$ $(a > 0)$,

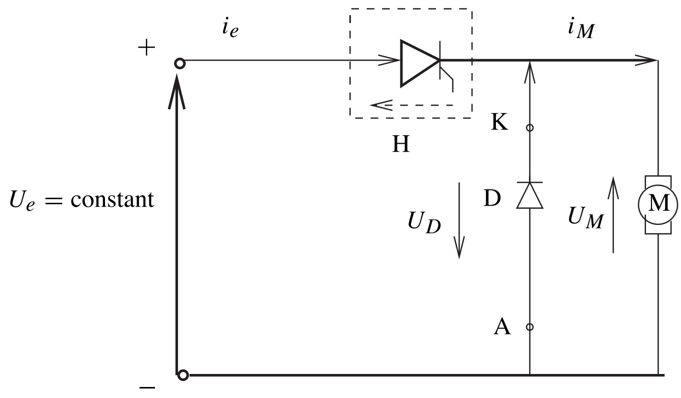
then we have an infinity of transitions in a finite time. This means that we have a Zeno behavior.

2.2 Chopper – a multimode system

An example which can be modeled as a hybrid system comes from the power electronics, where it is often used.

Consider an electrical mode-switching system containing fast switching components, such as diodes, thyristors combined with continuous dynamic components such as inductors, capacitors and electrical motors. In a very detailed model, even the transitions of the switching components would be modeled as continuous equations. In a more practical model, the transitions of the fast switching components would be idealized to instantaneous transitions.

One concrete example of such an electrical system is the chopper. We use the principle of chopper when we want to vary a continuous current coming from a source of constant continuous voltage. This principle consists of starting and stopping the load periodically. The load could be, for example, an engine of continuous current M represented by the resistance R, the inductance L and the electropower E_m.



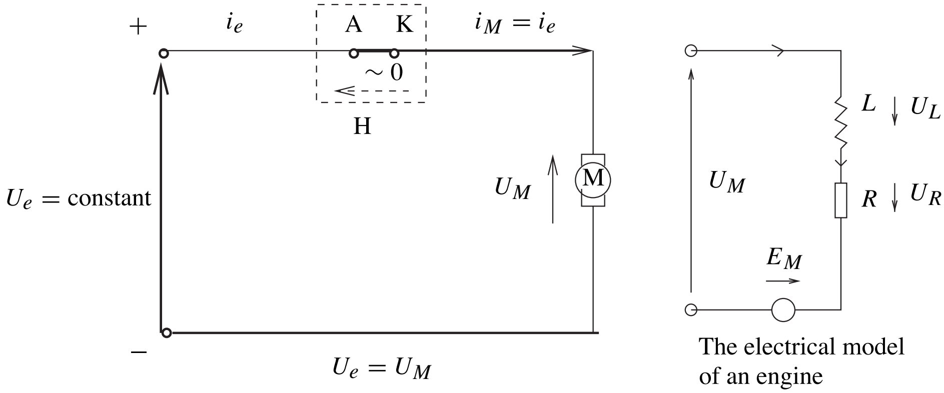
The first phase. Let us suppose that the switch is closed. The chopper H enters in conduction and allows the passing of the current i_e. Its "equivalent" circuit is a short-circuit. As the diode D accepts the passing of the current only from anode to the cathode, i_e must follow the way of the engine M, $i_e = i_M$ and we have in terms of voltage $U_e = U_M$. The engine receives all the power. We have the equation for the voltages:

$$U_M = U_R + U_L + E_m$$

and then in this case we obtain

$$U_e = Ri + L\frac{di}{dt} + E_m. \tag{2.1}$$

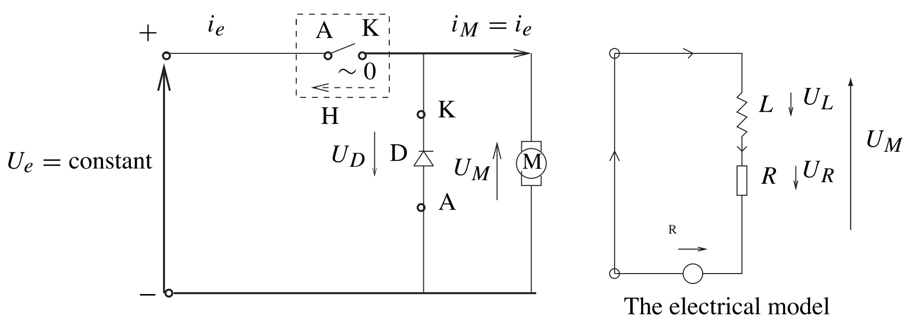
In a "steady" mode, the engine turns at a constant speed, inducing a constant current (there are always ripples of the current, but we take into consideration the average value) and then $U_L = L\frac{di}{dt} = 0$. In general, the resistance R of the engine's winding is considered small and we can suppose that U_R is also zero. The equation (2.1) gives then $U_M = E_m$.



The electrical model of an engine

If we let the chopper supplied all the time, then the engine will always turn at its maximum speed and it is not always desired. The problem is that if the chopper is cut, then we will have zero at the terminals of the engine. The goal is to obtain an average value of the voltage, $(U_M)_0$, controlled by the chopper at the engine's terminals.

The second phase. We open the switch (the chopper is blocked) in order to annulate the voltage at the terminals of the engine. At the time when the switch is opened, the current generated by the chopper becomes instantaneously zero (theoretically), implying a strong decrease of the engine's voltage, U_M. If there had not been the diode, there would be an opposite voltage on the chopper, which is destructive. We have the Lenz's law, saying that the diode opposes to the effect which gave rise to its current. Thus the diode enters in conduction (short-circuit) and imposes a null voltage at the terminals of the engine, which implies $U_M = 0$. Then the equation governing



The electrical model of an engine

the system is the following:

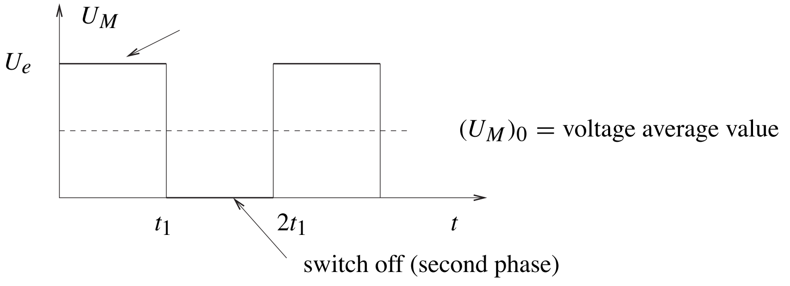
$$0 = U_R + U_L + E_m = 0, \quad \text{or}, \quad 0 = Ri + L\frac{di}{dt} + E_m. \tag{2.2}$$

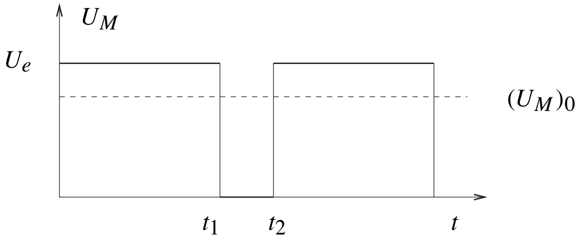
We close the switch of the chopper and the engine consumes again current. The diode remains blocked, because the current takes the way of the engine.

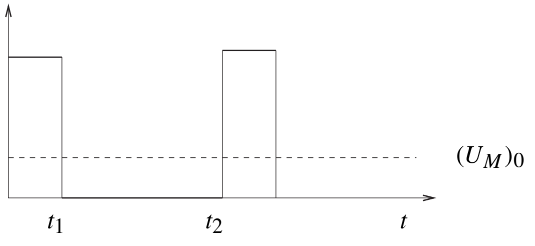
In order to have a smooth continuous current, we impose the load to be perfect, i.e. $L = \infty$.

This happens only theoretically. The consequence of the load's imperfection is that the current is not a piecewise constant function, but an exponential function. The equation (2.2) implies that the current i will decrease exponentially, $i = \exp(-\frac{t}{\tau})$ (where τ is a constant, the time of the system).

The commutations correspond to the passing from one phase of operation to the following one. Normally, they have a weak duration compared to the period of operation. In general, they are considered instantaneous.







Conclusion of the two phases for idealised instantaneous transitions:

As we see in the above figure, the voltage average value $(U_M)_0$ is related to the period of time after which we switch from one phase to another.

3 Hybrid systems and hybrid computing

The hybrid systems are the source of theoretical and computational difficulties. These difficulties can be summarized through the following points:

1. When a system is evolving, it is constrained by spatial and temporal constraints.
2. If the vector fields are nonlinear then, in general, it is impossible to build a closed form solution. Thus looking for the position of the representative point of the state relatively to the constraints is impossible.
3. On the other side, it is possible to build a closed form solution for linear systems.
4. The idea is then to use models built on piecewise dynamical systems. This is a classical idea in the modelling community (for example in the field of electrical circuits modelling).

We will make clear the methodology through two simple examples: the integration of ordinary differential equations and classical control theory. Two other examples regarding the study of plane dynamical systems and the generalized splines associated to hybrid systems can be found in [4].

3.1 Integration of ordinary differential equations.

A simple example will explain the idea (see also [7]).

A basic point in the definition of a hybrid system is that we have a decomposition of the *state* space, then we look at a dynamical system in a chart of the space. So we can imagine that a non linear differential equation is a (complicated) object that we will be approximated by piecewise (simple) linear differential equations. *Note the fact that at the opposite of the idea of numerical methods it is not the time space which is decomposed, but the space itself* (*Lebesgue*). This decomposition of the space induces a decomposition of the time. This is a key point in hybrid computing: same treatment for space and time variables.

Let us consider the initial value problem:

$$x'(t) = (x(t))^2,\ x(0) = 1$$

The solution of this IVP is:

$$X(t) = \frac{1}{1-t}$$

(moving singularity).

Let us consider a splitting of the *state* space (that is the x space) by a grid of mesh h (h being a positive real: the step). The subdivision is built of intervals $I_i = [ih, (i+1)h]$. The function $f(x) = x^2$ is defined on the state space and we will approximate f on each interval I_i by an affine function $g_i(x) = a_i x + b_i$. The simplest way to do that is by an interpolation process, using the extremities of the intervals. We obtain the following approximant:

$$g_i(x) = (2i+1)hx - i(i+1)h^2.$$

The solution of the (local) differential equation

$$x' = (2i+1)hx - i(i+1)h^2$$

is

$$x_i(t) = (ih - \frac{i(i+1)h}{2i+1}) \exp[(2i+1)h(t-t_i)] + \frac{i(i+1)h}{2i+1},$$

t_i is the value of t such that $x_i(t_i) = ih$. If we impose the continuity of the solution $x_i(t_i) = x_{i-1}(t_i)$ then we obtain

$$t_i - t_{i-1} = \frac{2}{h(2i-1)} \ln\left(\frac{i}{i-1}\right).$$

(Here we see the big difference between hybrid and the classical numerical approaches). The summation of the series

$$T_\infty = \sum_{i \geq 1} (t_i - t_{i-1})$$

shows that $T_\infty \approx 1$. The experiment [9] shows the very good concordance between the true solution and the hybrid computed solution and also the very good behavior of the method compared with a Runge-Kutta method of the same order.

It is not very difficult to prove a first convergence result using classical hypothesis on the f function. For example we have the following result:

Theorem 3.1. *If f is k-lipschitz on a compact subset of the x-axis and if we use interpolant affine approximants for f, then the solution of the piecewise linear differential equation converges in the ∞-norm, on the compact, to the solution of the same IVP.*

This first result opens the way to new directions:

- As the convergence of the approximate solution to the true solution is of the same order as the convergence of f_h (the piecewise approximation) to f, can we improve the convergence by a better choice of the affine approximants ?
- Can we improve the convergence by using quadratic piecewise approximants?
- How can we compare these results to the Runge-Kutta classical convergence results?

We have done all these studies (see [9]) and the answers are positive. There is indeed a better choice of the interpolant approximant, for example the choice of the distributed error (following an idea of professor P.J.Laurent) is an excellent substitute to the interpolant and the experimental studies show that the method is a second order method. We can also use a framing of the function on an interval by two linear approximants.

It is also possible to use quadratic approximants (but we pay the price of more difficult computations).

Of course the first results of our method show several points which need more study: the computation (using Maple) is slower for our method than using a classical Runge-Kutta method with adaptative steps. The probable reason is that Runge-Kutta is compiled and very optimized. A more interesting point is related to the behavior of the solution; if the solution has a vertical asymptote then we have to use smaller t-steps and so the number of iterations is growing. This is a difficult point, but in these situations a classical numerical method has some difficulties.

Another problem with our method is related to the horizontal asymptotes. The point is that for such an asymptote (say x_0) we must have $f(x_0) = 0$. The Runge-Kutta method runs very well in these situations, but our method is very bad because we have to approximate a zero of f using piecewise approximants and, in general, we will miss the zero. We have several methods for handling this situation. The best one is probably to use a system of the form $x' = f(x),\ t' = 1$.

3.2 Optimal control [6]

In this subsection we will present another application of hybrid computing in the field of optimal control (for a basic presentation of the field of optimal control in hybrid systems see [2]). We describe the methodology through a classical example in the field.

An interesting and different optimal control approach for hybrid systems can be found in [5]. By solving an optimal control problem, the authors determine the conditions under which the system satisfies the desired properties.

We consider here two aspects of a classical problem. The first one is the following (the reachability problem): given a target, we determine the set of initial states such that, starting from here and applying different transitions between discrete modes, we can reach the target.

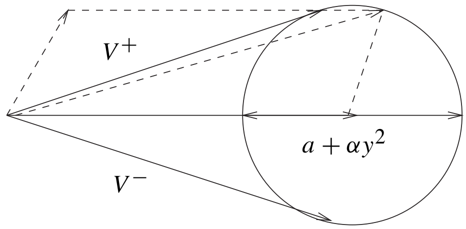
The second point is to solve a problem of optimal control, using Pontryagin's maximum principle.

Description of our problem (the Zermelo's problem): consider a river unbounded on the x-axis, represented by the strip $\{(x, y) \in \mathbb{R}^2; -b \leq y \leq b\}$ (b a positive constant), and a swimmer trying to reach a target (an isle) given by the equation $g(x, y) = (x-c)^2 + (y-d)^2 = r^2$. The velocity of the swimmer is a function of the angle θ made by his trajectory with the stream. There is a waterfall at a given distance from the origin. We consider the stream non-linear, equal to $(a + \alpha y^2, 0)$ (a, α given constants; α can be calculated from the condition that the stream is zero at the banks).

The system we consider is the following (it gives the coordinates of the swimmer):

$$\begin{cases} \frac{dx}{dt} = a + \alpha y^2(t) + \cos\theta(t) \\ \frac{dy}{dt} = \sin\theta(t) \end{cases}$$

a) By composing the two vectors, of the stream and of the swimmer's velocity, we obtain at each moment the region where he can swim:



It is sufficient to calculate the two trajectories given by the tangents, all the other possible trajectories will be between these two curves (Osgood-Montel's theorem).

Theorem 3.2. *Consider the differential equation:*

$$\dot{y} = f(x, y),$$

where f is continuous with respect to the set of variables y and x. The integrals starting from a point P lie between two limiting curves which are integrals starting from P: the higher integral and the lower integral.

The projections of the two tangents on the x-axis, respectively on the y-axis, give us the following differential systems:

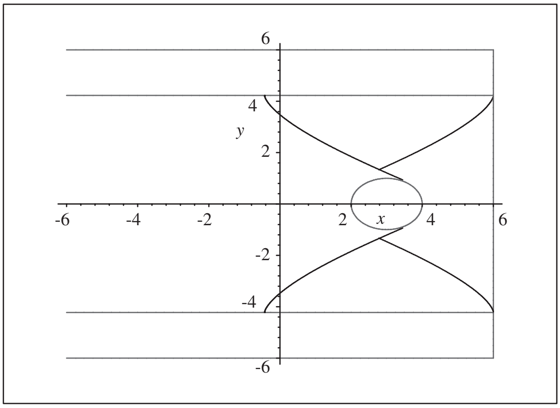
$$V^+ \quad \begin{cases} \frac{dx}{dt} = \frac{(a+\alpha y^2)^2-1}{a+\alpha y^2} \\ \frac{dy}{dt} = \frac{\sqrt{(a+\alpha y^2)^2-1}}{a+\alpha y^2} \end{cases},$$

$$V^- \quad \begin{cases} \frac{dx}{dt} = \frac{(a+\alpha y^2)^2-1}{a+\alpha y^2} \\ \frac{dy}{dt} = -\frac{\sqrt{(a+\alpha y^2)^2-1}}{a+\alpha y^2} \end{cases}$$

We integrate the tangent vectors and we obtain:

The two curves from the left are obtained by direct integration, the other two by inverse integration, as we know the last point of rescue. The two extreme strips represent the region where the stream becomes less than or equal to the velocity of the swimmer, which means that he can avoid the waterfall (he can swim upstream).

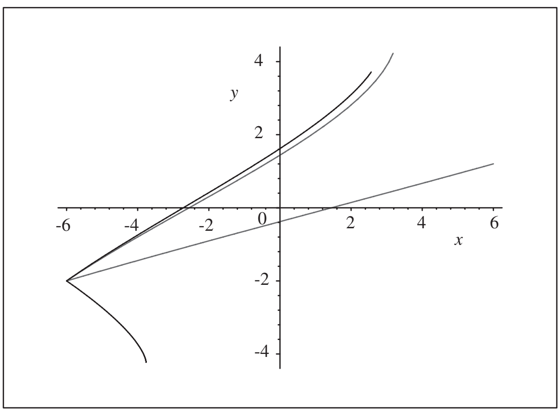
If the initial point belongs to region A (first from the left), the swimmer can reach the isle by different strategies. If he starts from regions B or D (above or below,



between the curves), he must enter into the extreme strips, go upstream and choose another departure point. The region C (from the right) gives no solution to avoid the waterfall. We have supposed that he cannot reach the boundaries (except for those of the extreme strips).

If we calculate the length of different trajectories starting from the region A which arrive on the isle, the length is almost the same (we consider that the swimmer does not enter in the two extreme strips).

We verified (for V^+) that for all the angles which vary between 0 and π, the trajectories calculated from the initial system with θ constant rest between the curves given by the tangents. If we consider the initial point (x_0, y_0) at the moment t_0, we have the equation $x - x_0 = \frac{\alpha}{3\sin\theta}y^3 + \frac{a+\cos\theta}{\sin\theta}y - \frac{y_0(\alpha y_0^2+3a+3\cos\theta)}{3\sin\theta}$:



b) We solve the non-linear differential system approximated by piecewise linear differential systems.

Let us consider a splitting of the state space on the y-axis by a grid of mesh h. On each interval $I_i = [ih, (i+1)h]$ of the subdivision, we approach the stream by the linear function $g_i(y) = m_i y + n_i$. Knowing the two points which define the line, $(a + \alpha(ih)^2, ih)$ and $(a + \alpha((i+1)h)^2, (i+1)h)$, we obtain:

$$g_i(y) = \alpha h(2i+1)y + a - \alpha i h^2(i+1).$$

We will assume that the system is controllable and we will be looking for the controls which effect a prescribed transfer of the system in minimal time.

We apply the Pontryagin's maximum principle for the linear case [3], on the strip $[ih, (i+1)h]$:

$$\begin{cases} \frac{dx}{dt} = g_i(y) + \cos\theta \\ \frac{dy}{dt} = \sin\theta \end{cases}$$

The Hamiltonian of the system is

$$\mathcal{H} = \lambda_x(g_i(y) + \cos\theta) + \lambda_y \sin\theta + 1,$$

which gives the Euler-Lagrange equations:

$$\dot{\lambda}_x = -\frac{\partial \mathcal{H}}{\partial x} = 0,$$

$$\dot{\lambda}_y = -\frac{\partial \mathcal{H}}{\partial y} = -\lambda_x \frac{\partial g_i(y)}{\partial y}.$$

θ minimizes the Hamiltonian and then $\tan\theta$ can be calculated:

$$0 = \frac{\partial \mathcal{H}}{\partial \theta} = -\lambda_x \sin\theta + \lambda_y \cos\theta \Rightarrow \tan\theta = \frac{\lambda_y}{\lambda_x}.$$

We replace $\lambda_y = \lambda_x \tan\theta$ in the form of $\mathcal{H}$ and we equal this to zero. Thus we obtain:

$$\begin{cases} \lambda_x = -\frac{\cos\theta}{1+g_i(y)\cos\theta} \\ \lambda_y = -\frac{\sin\theta}{1+g_i(y)\cos\theta} \end{cases}$$

and then $\frac{\cos\theta}{1+g_i(y)\cos\theta}$=constant.

We may now substitute the derivative of λ_x and λ_y with respect to the time into either $\dot{\lambda}_x = -\frac{\partial \mathcal{H}}{\partial x}$ or $\dot{\lambda}_y = -\frac{\partial \mathcal{H}}{\partial y}$ and use the fact that $\frac{dy}{dt} = \sin\theta$ to obtain

$$\frac{d\theta}{dt} = -\cos^2\theta \frac{\partial g_i(y)}{\partial y}.$$

We wish to find the minimal-time path from a certain point $(x_0, y_0 = ih)$ to a final point $(x_f, y_f = (i+1)h)$. We consider θ_f the final heading angle. Since λ_x is constant, we can write that

$$\frac{\cos\theta}{1 + g_i(y)\cos\theta} = \frac{\cos\theta_f}{1 + g_i(y)\cos\theta_f}$$

and using the form of $g_i(y)$, we obtain

$$\frac{dy}{d\theta} = -\frac{\sec\theta\tan\theta}{(2i+1)\alpha h}.$$

Since we have calculated $\frac{d\theta}{dt}$, we can obtain now $\frac{dx}{d\theta} = \frac{dx}{dt} \cdot \frac{dt}{d\theta}$. We have the following system:

$$\begin{cases} \frac{dx}{d\theta} = \frac{1}{(2i+1)\alpha h}[-\sec\theta - \sec^2\theta(\sec\theta_f + a + \alpha i^2 h^2) \\ \qquad\qquad + \sec^3\theta] \\ \frac{dy}{d\theta} = -\frac{\sec\theta\tan\theta}{(2i+1)\alpha h} \end{cases}$$

which we integrate and obtain:

$$\begin{cases} x - x_f = \frac{1}{(2i+1)\alpha h}(-\frac{1}{2}\ln\frac{\sec\theta+\tan\theta}{\sec\theta_f+\tan\theta_f} + \frac{1}{2}(\tan\theta\sec\theta \\ -\tan\theta_f\sec\theta_f) - (\sec\theta_f + a + \alpha i^2h^2)(\tan\theta - \tan\theta_f)) \\ y - y_f = \frac{1}{(2i+1)\alpha h}(\sec\theta - \sec\theta_f) \end{cases}$$

As we know the initial (x_0, y_0) and final (x_f, y_f) points, this system gives us the angles θ_0 and θ_f. We solve numerically this system and we choose the real solutions. It is convenient to use θ as the independent variable instead of t in the equation $\frac{d\theta}{dt} = -\cos^2\theta\frac{\partial g_i(y)}{\partial y}$. Thus we have $\frac{dt}{d\theta} = -\frac{1}{\cos^2\theta} \cdot \frac{1}{(2i+1)\alpha h}$ and obtain the time to go between the two points:

$$t - t_f = -\frac{1}{(2i+1)\alpha h}(\tan\theta - \tan\theta_f).$$

An interesting problem is to generalize the method for time-optimal trajectories (both in the constant and linear case) for more strips. Thus we could calculate the optimal trajectory to the isle, starting from any initial point (of the reachability set).

The idea for more strips is the following: we know the initial point (x_0, y_0) and the final point (x_f, y_f). We already have a splitting on the Oy-axis, consider that between y_0 and y_f there are $(m+1)$ points: $y_0 < y_1 < \cdots < y_m = y_f$. Take now a splitting of the interval $[x_0, x_f]$ on the Ox-axis, containing $(n+1)$ points: $x_0 < x_1 < \cdots < x_n = x_f$.

Suppose that $y_0 = ih$. On the horizontal strip $[ih, (i+1)h]$, consider all the trajectory between $[x_0, y_0 = ih]$ as initial point and $[x_k, (i+1)h]$, $k = \overline{1, n}$ as final point and retain the time-optimal one. As starting point for the next strip we consider the point on the line $y = (i+1)h$ for which the minimum time was obtained, say $[x_l, (i+1)h]$.

We follow the same process on the following strip $[(i+1)h, (i+2)h]$, taking in consideration as presumably final points only those whose abscissa is strictly bigger than x_l. And so on until we reach the point (x_f, y_f).

Further generalizations of the Zermelo's problem are easy to develop. For example, let the banks of the river be represented by some continuous curves (preferably with a continuous derivative) rather than by straight lines, assume a velocity field of the general type $u = u(x, y)$, $v = v(x, y)$ and leave the point of departure and point of arrival unspecified. Then we have a general problem, which enter into the category of "free-endpoint problems".

4 Conclusion

Even if the concept of hybrid systems exists for a long time as the examples of this paper (or other examples from various fields [6]) show, there is an important study in the future for the theoretical aspect of hybrid systems.

The aim of this paper is to give a description of hybrid systems concept from a mathematical point of view. Roughly speaking, the mathematical theory of hybrid systems is an answer to the question of building a sought framework for analyzing systems where two distinct dynamics are in interaction: an analog or continuous dynamics which is supposed to model physical process (dynamics which is represented by a family of continuous vector fields) and on the other side a digital or discrete dynamics, which is the model of a digital device or a program acting on the analog agents. While continuous dynamics have been studied in control theory, and discrete systems have been investigated in computer science, the combination of complexity in both discrete and continuous aspects leads to problems that are not well understood.

We started by presenting the motivation for hybrid systems modeling. This motivation comes from the large number of applications in different domains which can be modelled as hybrid systems, as traffic control systems, electric power circuits, economic models, chemical reactions or cell-cycle control system in biology. We presented some important aspects in the theory, as the system trajectory and the construction of the transition multiapplication.

In order to better understand the behavior of a hybrid system, we illustrated the concept through two examples coming from two different fields. These examples are simple enough to be analyzed by "hand", but they will mark some limits of the hybrid model.

We remarked the necessity of introducing a new computing method adapted to this complex class of systems, method that we called hybrid computing. The presentation of this method was given through two examples: the integration of ordinary differential equations and a classical problem from control theory.

The method of hybrid computing gives the ability to create new algorithms which are sometimes more accurate than the classical numerical algorithms. The concept of approximation underlying hybrid computation seems to be a key point for future work in computer algebra.

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This volume contains nine refereed research papers in various areas from combinatorics to dynamical systems, with computer algebra as an underlying and unifying theme.

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